MEMOIRS
OF THE
AMERICAN MATHEMATICAL SOCIETY

NUMBER 53

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Library of Congress Catalog Number 52-42839
International Standard Serial Number 0065-9266
International Standard Book Number 0-8218-1253-X

Third printing, 1979

Printed in the United States of America

JORDAN ALGEBRAS OF SELF-ADJOINT OPERATORS

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1. Introduction. In this paper we present a real non-commutative counterpart to the theory of von Neumann algebras. The earliest evidence of the existence of such a theory is found in von Neumann's paper: On an algebraic generalization of the quantum mechanical formalism (Part I), Mat. Sborn.1(1936), 415-484, although fragments of the subject can be found in the writings of numerous other authors.

The objects studied in the present paper are weakly closed Jordan algebras of bounded self-adjoint (s.a.) operators, herein referred to as JW-algebras. After some preliminary terminology in section 2, we settle on a notion of "commutativity" (section 3) and this allows us to define the center of a Jordan algebra of s.a. operators as the totality of operators therein commuting (under the ordinary operator product) with each operator in the algebra. The initial portion of the investigation is concerned with showing that the projections in a JW-algebra form a complete orthomodular lattice. For this purpose, annihilators are introduced in section 6, setting the stage for isolation of the type I summand in section 7. Preparatory to a characterization of annihilators, we introduce quadratic ideals (section 5), these being linear subspaces invariant under all quadratic maps of the form $a \rightarrow aba$. These ideals play a role similar to that of left and right ideals in an associative operator algebra.

Having split off the type I portion of an arbitrary JW-algebra, we proceed to develop a theory of relative dimensionality which permits further classification into "finite" and "infinite" parts. In a von Neumann algebra, two projections e and f are said to have the same relative dimension (i. e. relative to the containing algebra) if there is a partial isometry x in the algebra with $x^*x = e$ and $xx^* = f$. If, however, we start with a JW-algebra A and ask for a partial isometry in the complexification $A + iA$ linking e and f, our program fails unless $A + iA$ is already a von Neumann algebra, since A is stable under conjugation by unitaries from $A + iA$ only in this case. On the other hand, any JW-algebra A is invariant under conjugation by

Received by editors July 20, 1964 and, in revised form, September 1, 1964.

* Most of this work was completed while the author held a NATO Postdoctoral fellowship.

symmetries (= s.a. unitaries) from A, so it is reasonable to agree that two projections e and f have the same dimension relative to A if there is a symmetry s in A which exchanges e and f, that is, $ses = f$. Since symmetry exchange is not transitive in general, we take the minimal transitive coarsening of symmetry exchange as the basis for our dimension theory. Thus equality of dimension ("equivalence") for two projections e and f relative to A is defined to mean that by a finite number of symmetry exchanges, with all symmetries involved belonging to A, one can get from e to f. Sections 8, 9 and 10 relate our notion of equivalence to perspectivity and to Dixmier's concept of "position p'".

In section 11 we show that our equivalence relation bears the correct relation to the center of the algebra, and that even though equivalence is not additive in general, enough additivity remains to carry out the usual dimension-theoretic arguments with minor modifications.

One of the fundamental results of the paper, the Comparison Theorem, is obtained in section 12. For a JW-algebra A whose center consists only of real multiples of the identity (i.e. a factor), the force of this theorm is that two projections can always be "compared", in the sense that one is equivalent to a subprojection of the other. In fact, our techniques show that this comparison can be effected by a single symmetry, thus increasing the efficiency of the comparison. Analogous results are obtained for algebras with centers.

We take up the question of "finiteness" in section 13. From von Neumann's theory of continuous geometry [17] it is reasonable to associate "finiteness" with modularity of the projection lattice (see section 2 for terminology). Pursuing this theme, we relate modularity to two notions of "finiteness" involving the equivalence relation. A projection is called modular if the principal ideal it generates in the projection lattice is a modular lattice. The key result of this section says that the lattice join of two modular projections is also modular. An important consequence of this latter fact is that on the set of modular projections, the relations of perspectivity and symmetry exchange coincide and are transitive (Corollary 21).

Further examination of modularity properties in section 14 leads to a type decomposition similar to the classical one for a general von Neumann algebra. One notable feature of our type classification is that it coincides with the usual one when our algebra happens to be the set of s.a. operators in a von Neumann algebra. In this case, our concept of equivalence is the same as perspectivity and unitary equivalence.

Section 15 capitalizes on the dimension theory developed earlier, especially the Comparison Theorem, and most of the standard structure theory for von Neumann algebras is carried over, with appropriate modifications, to JW-algebras.

In sections 16 and 17 some recent results of Arlan Ramsay are exploited to construct dimension functions. As mentioned earlier, our equivalence relation lacks additivity. We show how to circumvent this apparent fault by coarsening the equivalence so that it becomes completely additive. In case our algebra acts on a separable Hilbert space and has no type III portion, we can accomplish this coarsening in one and only one way.

In section 18, JW-algebras having modular projection lattices are characterized as those which possess a faithful completely additive symmetry invariant center-valued trace.

Finally, we present an example of a new factor phenomenon which does not occur in the von Neumann theory (section 19). This factor is "discrete", of "finite class", and yet is infinite dimensional as a real linear space.

Special thanks are due Arlan Ramsay for many helpful discussions on the subject matter of this paper and for making available to the author the contents of [18] prior to publication. We are also grateful to Professor Irving Kaplansky for critically reading the first draft and suggesting a number of improvements in the exposition.

Our debt to the writings of Dixmier [2, 3, 4, 5] and Kaplansky [10, 11, 12, 13] is considerable.

2. Terminology. All operators in this paper are understood to be bounded. By a JW-algebra we shall mean a real linear space of s.a. operators which is closed under the operation of squaring and also closed in the weak operator topology.

The (normalized) Jordan product is $a \circ b = (ab + ba)/2$, where juxtaposition denotes ordinary operator multiplication. If A is a uniformly closed Jordan algebra of s.a. operators, then by the Spectral Theorem, A contains, along with $a \in A$, such continuous functions of a as $|a| = (a^2)^{1/2}$ and $a^{1/2}$ if $a \geq 0$. Also fundamental is the formula: $aba = 2\,(a \circ (a \circ b)) - a^2 \circ b$ which shows that $aba \in A$ if $a, b \in A$.

For future reference, we remind the reader that a lattice L is called modular if $(e \cup f) \cap g = e \cup (f \cap g)$ whenever $e \leq g$, for all $e, f, g \in L$.

For basic facts concerning von Neumann algebras we refer the reader to the book of Dixmier [2].

3. Commutativity and the center. First we recall that in the language of Jordan algebra theory, a and b are said to "operator commute" (the terminology is a bit unfortunate here) if their Jordan translations J_a and J_b commute as linear operators, where $J_a(x) = a \circ x$. Stated otherwise, this means that $a \circ (b \circ c) = (a \circ c) \circ b$ for all elements c of the Jordan algebra under consideration.

PROPOSITION 1. *For s.a. operators a and b the following are equivalent:*

1) $ab = ba$.
2) *a and b "operator commute" relative to any Jordan algebra of s.a. operators containing them.*
3) $a^2 \circ b = aba$.

In particular, a Jordan algebra of s.a. operators consists of pairwise commuting operators if and only if it is associative relative to the Jordan product.

PROOF. The implication 1) $\Rightarrow$ 2) is clear. 2) $\Rightarrow$ 3): From $a \circ (b \circ c) = (a \circ c) \circ b$ we get (with $a = c$): $a^2 \circ b = a \circ (a \circ b)$. But $a \circ (a \circ b) = (a^2 \circ b + aba)/2$. 3) $\Rightarrow$ 1): The equation $a^2 \circ b = aba$ is equivalent to $a(ab - ba) = (ab - ba)a$. From Kleinecke's proof [14] of Kaplansky's conjecture, we see that $z = ab - ba$ is topologically nilpotent. But z is skew-adjoint and iz is self-adjoint with the same spectral radius as z, namely zero. Thus $z = iz = 0$ and $ab = ba$.

Let A be a Jordan algebra of s.a. operators. We shall say that A is *commutative* (or *abelian*) if the operators in A commute (under ordinary operator multiplication). The *center* of A is the intersection of all maximal abelian Jordan subalgebras of A. We call A a *factor* if $1 \in A$ and the center of A consists of all real multiples of the identity.

4. JW-algebras. The squaring map $a \to a^2$ on the s.a. operators is known to be discontinuous in both the weak and strong operator topologies. We pause therefore to verify

LEMMA 1. *Let A be a Jordan algebra of s.a. operators. Then the weak closure of A is a JW-algebra.*

PROOF. It is well-known [4] that the weak and strong closures of a convex set of operators coincide.

Let b lie in the weak closure of A. Then b is also a strong limit point

of A. For $a \in A$, and x a vector in the underlying Hilbert space, $|\,\|ax\| - \|bx\|\,| \leq \|(a-b)x\|$ so if a approaches b strongly, $\|ax\|$ and $\|bx\|$ are close, as are $\|ax\|^2$ and $\|bx\|^2$. That is to say, however, that $(a^2x|x)$ and $(b^2x|x)$ are close, so that a^2 approaches b^2 weakly.

LEMMA 2. Let A be a JW-algebra. Then A contains, along with each of its operators a, the self-adjoint part of the double commutant $\{a\}''$ of a (or a maximal ideal thereof). In particular, A contains the spectral projections and the range projection of each $a \in A$.

PROOF. Let S = all s. a. operators. For $a \in A$, $\{a\}'' \cap S$ is just the weak closure of the algebra of real polynomials, with constant term, in a. If $1 \in A$, we are through, for then $\{a\}'' \cap S \subset A$. If $1 \notin A$, then A contains the maximal ideal in $\{a\}'' \cap S$ of weak limits of real polynomials in a without constant term. In either case, the Spectral Theorem now tells us that the spectral projections of a lie in A as does the range projection.

COROLLARY 1. Any JW-algebra is uniformly generated by its projections.

Preliminary to a more serious examination of the situation, we now show that the projections in a JW-algebra "almost" form a complete lattice.

THEOREM 1. Let A be a JW-algebra and let $\{e_i\}$ be any orthogonal set of projections in A. Then $e = \mathrm{LUB}\ e_i$ is in A. Further, if e and f are two projections in A, then $e \cap f$ is in A.

PROOF. Given projections $e, f \in A$. A classical result of von Neumann (see [7] for an account and a generalization) says that the sequence $(efe)^n$ converges to $e \cap f$ in the strong operator topology. However, $efe = 2(e \circ (e \circ f)) - e \circ f \in A$ so we have $e \cap f \in A$. Note that if we work with the algebra $A_1 = \{\alpha + a : a \in A,\ \alpha \text{ real}\}$ (also weakly closed) then we conclude that $e \cup f = 1 - ((1-e) \cap (1-f)) \in A_1$ and hence that the projections form an orthocomplemented lattice. It further follows that the projection lattice of A_1 is complete, since given a set $\{e_i\}$ of projections in A_1, we can first take finite suprema to get a directed family of projections in A_1 which must then converge weakly and strongly to $e = \mathrm{LUB}\ e_i$. However, we prefer to get completeness later from other considerations.

For an orthogonal family $\{e_i\}$ of projections in A we first take finite suprema (=finite sums) to get a directed collection in A having the same LUB as $\{e_i\}$ and converging weakly and strongly to this LUB.

In § 6 we develop the annihilator machinery from which we may easily conclude that the projections in a JW-algebra form a complete orthocomplemented lattice, and in particular, that such an algebra contains a largest projection.

5. Quadratic ideals. Throughout this section only, A will be uniformly closed Jordan algebra of s.a. operators with $1 \in A$. A linear subspace I of A is called an absolute order ideal if $|a| \in I$, whenever $a \in I$ and $0 \leq b \leq a$ with $a \in I$ and $b \in A$ implies $b \in I$. The positive and negative parts of a s.a. operator are defined by $a^+ = (|a| + a)/2$ and $a^- = (|a| - a)/2$ respectively. A quadratic ideal is a linear subspace I of A with $aba \in I$ whenever $a \in I$ and $b \in A$.

The following is a non-commutative generalization of a result of Stone [20] concerning lattice and algebra ideals in C(X).

THEOREM 2. Let A be a uniformly closed Jordan algebra of s.a. operators with $1 \in A$ and let I be a linear subspace of A. If I is an absolute order ideal, then it is also a quadratic ideal. A uniformly closed quadratic ideal is an absolute order ideal.

PROOF. Suppose first that I is an absolute order ideal and take $a \in I^+$, $b \in A^+$. Normalizing, we can assume $0 \leq a, b \leq 1$. Then $0 \leq aba \leq a^2 \leq a$ so that $aba \in I$. For $b \in A$ arbitrary, $b = b^+ - b^-$ and if $a \in I^+$ we have ab^+a, $ab^-a \in I$, so $aba = ab^+a - ab^-a \in I$. Note also that for $a \in I$, $b \in A$ we have $|a|b|a| \in I$. Now for arbitrary $a \in I$, $b \in A$, write $a = a^+ - a^-$, $b = b^+ - b^-$. Then $aba = 2(a^+b^+a^+ - a^+b^-a^+ + a^-b^+a^- - a^-b^-a^-) - |a|b|a|$ and each member of the right hand side is in I. Thus $aba \in I$.

For the last statement, let I be a uniformly closed quadratic ideal. Since $1 \in A$ we have $a^2 \in I$ if $a \in I$. Hence $|a| \in I$ if $a \in I$. It remains to show that $0 \leq b \leq a$ and $a \in I$ imply $b \in I$. Note first that we have $a^{1/2} \in I$ if $a \in I^+$. By Lemme 2 of [2] (p.11) there is an operator c (not necessarily s.a. or in A— but this causes no difficulty) with $b^{1/2} = c\, a^{1/2}$. We estimate:

$$\| b^{1/2} - b^{1/2} a^{1/2} (1/n + a^{1/2})^{-1} \| = \| b^{1/2} (1 - a^{1/2} (1/n + a^{1/2})^{-1}) \|$$
$$= \| b^{1/2} (1 + n a^{1/2})^{-1} \| = \| c\, a^{1/2} (1 + n a^{1/2})^{-1} \| \leq \| c \| . \| a^{1/2} (1 + n a^{1/2})^{-1} \|$$
$$= (\| c \| / n) \| 1 - (1 + n a^{1/2})^{-1} \| \leq (\| c \| / n) \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Hence

$b^{1/2} a^{1/2} (1/n + a^{1/2})^{-1} \to b^{1/2}$ uniformly as $n \to \infty$. By symmetry $(1/n + a^{1/2})^{-1} a^{1/2} b^{1/2} \to b^{1/2}$ uniformly as $n \to \infty$, and therefore $a^{1/2} (1/n + a^{1/2})^{-1} b (1/n + a^{1/2})^{-1} a^{1/2} \to b$ uniformly as $n \to \infty$. Since the left member belongs to I, we have $b \in I$.

Quadratic ideals play a role in Jordan algebras similar to that played by left ideals in associative algebras.

Note that for any $a \in A$, $I = aAa$ is always a quadratic ideal. It is easily checked that the quadratic ideal property persists under passage to the uniform closure.

PROPOSITION 2. Let A be a uniformly closed Jordan algebra of s.a. operators with $1 \in A$. Then for $a \in A$, the following are equivalent:

1) $1 \in aAa$.
2) a^{-1} exists in A.
3) $A = aAa$.

Thus a^{-1} exists if and only if a belongs to no proper quadratic ideal. Every (proper) quadratic ideal is contained in a maximal one, and the intersection of all maximal quadratic ideals is zero.

PROOF. 1) $\Rightarrow$ 2) : If $1 = aba$, then a is invertible and $a^{-1} = ab = ba$.

2) $\Rightarrow$ 3) : For $b \in A$, $b = a(a^{-1} b a^{-1}) a$.

3) $\Rightarrow$ 1) is trivial.

The second assertion is clear, and the third is evident from a simple application of Zorn's Lemma.

Finally, suppose $a \in M$, for all maximal quadratic ideals M of A. If $(1 + aba)^{-1}$ does not exist, then $1 + aba \in I$, for some proper quadratic ideal I. Enlarge I to a maximal quadratic ideal M. Then $1 + aba \in M$, and by assumption, $aba \in M$. Thus $1 = (1 + aba) - aba \in M$, a contradiction. Hence $(1 + aba)^{-1}$

exists for all $b \in A$, For $b = -\lambda^{-1}$, λ any non-zero real, we see that $(\lambda - a^2)^{-1}$ exists, and hence a^2 has zero spectrum. Thus $a^2 = 0$ and $a = 0$, since a is self-adjoint.

6. Annihilators. We now embark on a program which is essentially the Jordan counterpart of Kaplansky's theory of Baer rings [13].

LEMMA 3. *Let* a, b, x, y *be s.a. operators with* $0 \leq x \leq a$, $0 \leq y \leq b$ *and* $ab = 0$. *Then* $xy = 0$.

PROOF. $0 \leq a^{1/2} y a^{1/2} \leq a^{1/2} b a^{1/2} = 0$ so $ay = 0$. But

$$0 \leq y^{1/2} x y^{1/2} \leq y^{1/2} a y^{1/2} = 0 \text{ so } xy = 0.$$

For any subset M of a JW-algebra A, we define the annihilator of M to be the set $M^{\perp} = \{ a \in A : ax = 0, \text{ for all } x \in M \}$. The next theorem plays the role of the Baer axiom in [13].

THEOREM 3. *For any subset* M *of a* JW-algebra A, *we have* $M^{\perp} = eAe$, *for some projection* $e \in A$.

PROOF. First observe that since $a \to ax$ is weakly continuous, $\{x\}^{\perp}$ and therefore $M^{\perp} = \bigcap \{x\}^{\perp}$ $(x \in M)$ are weakly closed. Thus $M^{\perp}$ is a JW-subalgebra of A. It is also clearly an absolute order ideal by Lemma 3. Choose in $M^{\perp}$ (via Zorn's Lemma) a maximal orthogonal family $\{e_i\}$ of projections and write $e = \text{LUB } e_i$. By Theorem 1, $e \in M^{\perp}$. Since $M^{\perp}$ is a quadratic ideal, $eAe \subset M^{\perp}$.

Conversely, if $a \in M^{\perp}$, then we must show that $x = a - ae = 0$. Now $x^*x = a^2 - a^2e - ea^2 + ea^2e \in M^{\perp}$. Let p be the range projection of x^*x. By Lemma 2, $p \in M^{\perp}$; this also follows from the fact that x^*x and p have the same annihilator.

Now $(x^*x)e = 0$ and hence $pe = 0$. But then $pe_i = 0$ for all i, so $p = 0$ (otherwise $\{e_i\}$ would not be maximal); therefore $x^*x = 0$ and $x = 0$, showing that $M^{\perp} = eAe$.

COROLLARY 2. *A JW-algebra contains a largest projection.*

From now on we shall assume that this largest projection is the identity operator.

THEOREM 4. The projections in a JW-algebra form a complete orthocomplemented lattice satisfying the orthomodular identity (axiom (M) of [15]):

$$e \leq f \text{ implies } f = e \cup (f \cap (1-e)).$$

PROOF. Given any set $\{e_i\}$ of projections in A, we write $\{e_i\}^{\perp} = fAf$, for some projection $f \in A$. Assuming, as we may, that $1 \in A$, set $e = 1 - f$. We have $e_i f = 0$, whence $(1-f)e_i = e_i$ so that $e_i \leq e$. Also if $e_i \leq h$, h a projection, then $e_i(1-h) = 0$ so $1-h \in fAf$. Thus $(1-h)e = 0$ and $e \leq h$, so $e = \text{LUB } e_i$.

For $e \leq f$ in A, $f \cap (1-e) = f(1-e) = f - e$. Since $e(f-e) = 0$, we have $e \cup (f \cap (1-e)) = e + (f-e) = f$.

We are now able to give a more complete analysis of annihilators in

PROPOSITION 3. In a JW-algebra A, the following are equivalent:

1) I is a weakly closed quadratic ideal.
2) I is a weakly closed absolute order ideal.
3) $I = M^{\perp}$, for some set $M \subset A$.
4) $I = eAe$, for some projection $e \in A$.

PROOF. 3) $\Rightarrow$4) by Theorem 3. 4) $\Rightarrow$3) is clear since $eAe = \{1-e\}^{\perp}$. Equally evident are the implications 4) $\Rightarrow$1) and 4) $\Rightarrow$2). By Theorem 2, 1) and 2) are equivalent. It remains to show that 1) implies 4).

But a weakly closed quadratic ideal I is a JW-subalgebra of A for if $a \in I$, $aAa \subset I$; since $1 \in A$, $a^2 \in I$. Thus I contains a largest projection by Corollary 2 and reading the latter half of Theorem 3 (with I in place of M) we see that $I = eAe$.

Some elementary properties of annihilators are collected in

PROPOSITION 4. Let A be a JW-algebra. For a projection $e \in A$, $eAe = \{e\}^{\perp\perp}$. For two projections e and f in A, $\{e\}^{\perp} = \{f\}^{\perp}$ implies $e = f$ and $\{e\}^{\perp} \subset \{f\}^{\perp}$ if and only if $f \leq e$. For any projection $e \in A$, $\{1-e\}^{\perp} = \{e\}^{\perp\perp}$ and $\{e\}^{\perp\perp} \cap \{e\}^{\perp} = 0$. For any $a \in A$, $\{a\}^{\perp} = \{|a|\}^{\perp}$ and the range projection p of a is the unique projection in A for which $\{a\}^{\perp} = \{p\}^{\perp}$. For two subsets M and N of A the usual rules hold: 1) $M \subset M^{\perp\perp}$; 2) $M \subset N$ implies $N^{\perp} \subset M^{\perp}$; 3) $M^{\perp\perp\perp} = M^{\perp}$; 4) the annihilator of a union is the intersection of the annihilators; and $M^{\perp\perp}$ is the smallest weakly closed quadratic ideal containing M.

The verification of these properties is routine and is left to the reader.

Finally, we consider some relations between annihilators and Jordan ideals and show that the technique of "dropping down" to a direct summand works for JW-algebras.

LEMMA 4. Let A be a Jordan algebra of s.a. operators with $1 \in A$. If M is any subset of A, then the Jordan ideal (M) generated by M is the (real) linear subspace spanned by operators of the form x^*mx where x is a finite product (in the associative sense) of operators from A, and $m \in M$.

PROOF. For $a_1, \ldots, a_n$, $b \in A$, $m \in M$ with $x = a_1 \ldots a_n$ it is enough to show that $b \circ (x^*mx)$ is a real linear combination of operators having the form mentioned. But $b(x^*mx) + (x^*mx)b = (x(1+b))^*m(x(1+b)) - (x^*mx + (xb)^*m(xb))$ and each member on the right has the desired form.

For a single operator $a \in A$, the principal Jordan ideal (a) generated by a is the linear span of operators of the form $x^*a\,x$, x a finite product from A.

PROPOSITION 5. Let A be a JW-algebra. The annihilator of a Jordan ideal in A has the form eAe with e a central projection. In particular, if e is any projection, eAe is a Jordan ideal if and only if e is central.

PROOF. If e is central and $x \in eAe$ we have $x = eae = ae = ea$. For $y \in A$, $y \circ x = e(y \circ x)e \in eAe$, so eAe is a Jordan ideal.

Now let I be a Jordan ideal. Then $I^{\perp} = eAe$ for some projection e by Theorem 3. Given $a \in A$, our task is to show that $ae = ea$ and this is accomplished if we show that $ea = eae$, or equivalently that $ea + ae \in \{1-e\}^{\perp} = eAe = I^{\perp}$. Set $y = ea + ae$ and take $x \in I$. We must show $xy = 0$. But $e \in eAe = I^{\perp}$, so $ex = 0 = xe$. Thus $xy = xae$ and $yx = eax$ so $xy + yx = xae + eax \in I$ by the ideal property. Hence $0 = e(xy + yx) = eax$ and $0 = (xy + yx)e = xae$ so $xy = xae = 0$ and e is indeed central. Now if eAe is a Jordan ideal, its annihilator $(1-e)A(1-e)$ is too, so that $1-e$ and e are central.

7. Type I. A will denote a JW-algebra throughout. Although this section is a "symmetrized" version of Kaplansky's theory of Baer rings in [13], there are nevertheless a few computational tricks which do not go without saying.

LEMMA 5. *The annihilator of a central subset is of the form* eAe *with* e *a central projection.*

PROOF. Let M be a central subset of A. We know that $M^{\perp} = eAe = \{e\}^{\perp\perp} = \{1-e\}^{\perp}$ for a unique projection e by Theorem 3 and Proposition 4. For any $a \in A$ we must show $ea = ae$ or what amounts to the same, $ea = eae$. In other words, our task is to show $ea(1-e) = 0$ and for this it is enough to verify that $ea + ae \in \{1-e\}^{\perp}$. Set $y = ea + ae$ and take $x \in M$. We are to show $xy = 0$. But $e \in \{e\}^{\perp\perp} = M^{\perp}$, so $ex = 0 = xe$. Thus $xy = xea + xae = xae$. Since x is central, $xae = axe = 0$, and the proof is finished.

Most of the remaining facts in this section are analogs of well-known facts found in Chapter I of [13]. The omitted proofs are left to the reader.

COROLLARY 3. *The annihilator of a central subset is a direct summand.*

COROLLARY 4. *The center* Z *of* A *has the Baer property (Theorem 3)*

LEMMA 6. *Let* $a \in A$, $\{e_i\}$ *a family of projections with* $e = \text{LUB } e_i$ *and suppose* $e_i a = 0$, *for all* i. *Then* $ea = 0$.

COROLLARY 5. *Let* $e \in A$. *Then there is a smallest (in the order of the projection lattice of the center) central projection* e *satisfying* $ea = a$.

For $a \in A$, we define the *central cover* of a, denoted $C(a)$, to be the smallest central projection e such that $ea = a$. If $C(a) = 1$, a is said to be *faithful*.

LEMMA 7. *Let* e *be a central projection*, $a \in eAe$. *Then* $C(a)$ *is the same as the central cover of* a *computed in* eAe.

LEMMA 8. *The central cover* $C(a)$ *of* a *is characterized as the unique projection (central by Proposition 5, unique by Proposition 4)* h *for which* $(a)^{\perp} = \{h\}^{\perp}$.

PROOF. Write the annihilator of the principal Jordan ideal (a) generated by a as $(a)^{\perp} = (1-h)A(1-h) = \{h\}^{\perp}$, with h central (Proposition 5). Since $a(1-h) = 0$ we have $C(a) \leq h$. Setting $g = 1 - C(a)$ we have $ga = 0$. Now if $x = a_1 \ldots a_n$ with each $a_i \in A$, $(x^*ax)g = x^*(ag)x = 0$ so $g \in (a)^{\perp}$ (Lemma 4) and $g \leq 1-h$ or $h \leq C(a)$. Thus $h = C(a)$.

LEMMA 9. *For projections* e *and* f *in a JW-algebra* A, *the following are equivalent:*

1) $C(e) \perp C(f)$; 2) $C(e) \perp f$; 3) $e \perp C(f)$;
4) $e \in (f)^{\perp}$; 5) $f \in (e)^{\perp}$; 6) $e(a_i \ldots a_n)f = 0$

for each finite sequence $a_1, \ldots, a_n \in A$.

Moreover, for $a \in A$, $z \in Z$ *we have* $az = 0$ *if and only if* $C(a)z = 0$.

PROOF. By Lemma 4, the principal Jordan ideal (e) generated by e consists of finite linear combinations of operators of the form $x^*e\,x$, where $x = a_1 \ldots a_n$ with each $a_j \in A$. The results follow at once from Lemma 8 and this observation.

A projection e is *abelian* (with respect to A) if eAe is commutative.

LEMMA 10. *Let* e *be a central projection. Then*

1) *If* f *is an abelian projection with respect to* A, *the projection* ef *is abelian with respect to* eAe.

2) *If* $g \in eAe$ *is an abelian projection with respect to* eAe, *then* g *is abelian with respect to* A.

COROLLARY 6. *A projection which lies in a direct summand is abelian there if and only if it is abelian in the entire algebra.*

A JW-algebra A is *type I* if it has a faithful abelian projection. Alternatively, A is type I if every summand contains an abelian projection. Obviously type I is preserved under the taking of LUB's of central projections. We sum up these remarks in

THEOREM 5. *In any JW-algebra* A, *there is a central projection* e *such that* eAe *is type I and* $(1-e)A(1-e)$ *contains no abelian projections other than zero. This projection is unique, and is the largest central projection* f *such that* fAf *is type I.*

8. *Some algebraic facts.* In the sequel we shall consider several reflexive, symmetric relations on the projections in a JW-algebra, most of which are not transitive in general. An examination of the connections between these relations will provide us with an effective tool for studying the projection geometries of JW-algebras.

Let A be a JW-algebra and let $e, f \in A$ be projections. A symmetry (= s.a. unitary) $s \in A$ is said to *exchange* e and f if $ses = f$. Dixmier [3]

showed that in order for there to be a symmetry s exchanging e and f it is necessary and sufficient that $\dim(e \cap (1-f)) = \dim((1-e) \cap f)$. Our interest, however, will be centered around the need to know when the symmetry s can be chosen to lie in the given JW-algebra A, keeping in mind that the usual double commutant technique is unavailable.

If s is a symmetry exchanging e and f and if we set $g = 1/2(1+s)$, then g can be thought of as the "angle bisector" of e and f. In fact if we let $C(e,f) = 1-e-f+ef+fe$, we obtain the "closeness operator" (an infinite dimensional analog to the square of the cosine of the angle between two subspaces) of Davis [1] and we immediately compute: $C(e,g) = C(f,g)$.

By a <u>partial symmetry</u> we shall mean an s.a. operator whose square is a projection. Thus if s is a partial symmetry with $s^2 = p$, then s^+ and s^- are orthogonal projections with sum p. Any operator $a \in A$ has a <u>canonical real polar decomposition</u> $a = s|a|$ with $|a| = (a^2)^{1/2}$ and $s = RP(a+) - RP(a^-)$ ("RP" denotes the range projection) a partial symmetry in A.

PROPOSITION 6. <u>Let</u> A <u>be a JW-algebra with projections</u> e <u>and</u> f <u>in</u> A. <u>If</u> $s \in A$ <u>is a partial symmetry satisfying</u> $ses = f$ <u>and</u> $sfs = e$ <u>then</u> s <u>can be extended to a symmetry</u> $t \in A$ <u>exchanging</u> e <u>and</u> f. <u>Moreover, for any two projections</u> $e, f \in A$ <u>there is always a symmetry</u> $s \in A$ <u>with</u> $s(efe)s = fef$.

PROOF. Let $p = s^2$. Then $e = sfs = s^2 e s^2 = pep$, so $e = pe = ep$. Similarly $f = pf = fp$ and hence $p \geqq e \cup f$. Set $t = s + (1-p)$. Then since s^+ and s^- are orthogonal projections in A with sum p, we have $s^3 = s$ and hence $t^2 = 1$. Since $(1-p)e = 0$ we have $tet = ses = f$.

For the second part, let $a = e + f - 1$ and note that in the canonical real polar decomposition $a = s|a|$ we have $|a| = c^{1/2}$, where $c = 1-e-f+ef+fe$ is the "closeness operator" mentioned above. Note also that e and f commute with $|a|$ and c, but not with a itself (if $ef \neq fe$, which is the only case of interest). Now $s(e+f)s = e+f$ and since s commutes with a, we have $s^2(e+f) = e+f$. Thus $s^2 \geqq e \cup f = RP(e+f)$. Also $|a| = sa = se - s(1-f)$ and hence $saf = sef$. But $s(efe)s = safes = |a|fes$ and by symmetry, $fes = fas = f|a|$ so that $|a|fes = |a|f|a| = fa^2 = fc = fef$. Moreover $s(fef)s = s^2(efe)s^2$ and since $s^2 \geqq e \cup f$ we have $s^2 e = e = es^2$ and $s(fef)s = efe$. Defining t as in the first part of the proof gives a symmetry with the desired property.

The norm distance between two projections e and f in a JW-algebra A can be a determining factor in the search for a symmetry $s \in A$ exchanging e and f.

PROPOSITION 7. *If e and f are projections in A with $\|e - f\| < 1$, then $ses = f$, where $s \in A$ is the symmetry defined by the formula*

$$s = c^{-1/2}(e + f - 1), \quad \textit{where}$$
$$c = 1-e-f+ef+fe \quad (\text{cf. Davis } [1], \S 4).$$

PROOF. As before let $a = e+f-1$ and apply the real polar decomposition to get $a = s|a|$. Then $a^2 = c$. Also $1-c = (e-f)^2$ and hence $\|1-c\| = \|e-f\|^2 < 1$ so that c^{-1} exists. The Spectral Theorem insures $s \in A$ and an easy computation gives $ses = f$.

A very simple condition for the existence of a symmetry exchanging two projections of a special kind is given in the next proposition.

PROPOSITION 8. *If s and t are anticommuting symmetries in A and $e = (1+s)/2$ and $f = (1+t)/2$ (projections in A) then $r = 2^{-1/2}(s+t)$ is a symmetry in A exchanging e and f.*

The computation is omitted.

A well-known and useful fact in the the theory of von Neumann and AW*-algebras says that for any two projections e and f, $e(1-f)$ has $(e \cup f) - f$ as its right projection and $e - (e \cap f)$ as its left projection ([10], Lemma 5.3).

In the self-adjoint Jordan theory, "left" and "right" projections merge into the range projection (denoted "RP") and we obtain

LEMMA 11. *For any two projections e and f in a JW-algebra A,*

1) $RP((1-f)e(1-f)) = (e \cup f) - f$.
2) $RP(e(1-f)e) = e - (e \cap f)$.

PROOF. Let $x = (1-f)e(1-f)$ and let $g = RP(x)$, $h = f + g$. Then $xf = 0$ implies $fg = 0$, since $\{x\}^{\perp} = \{g\}^{\perp}$. Thus h is a projection and our task is to show $h = e \cup f$. Of course $h \geqq f$. To prove $h \geqq e$, we observe $x(1-g) = 0$, so $0 = e(1-f)(1-g) = e(1-h)$. Suppose $k \geqq e \cup f$. Then $1-k$ annihilates both e and f, so $x(1-k) = 0$. Thus $(1-k)g = 0$ and $k \geqq g$. Together with $k \geqq f$, this gives $k \geqq h$. We have proved $h = e \cup f$, and $g = (e \cup f) - f$. Replacing e by $1-f$ and f by $1-e$ gives 2).

COROLLARY 7. <u>For any two projections</u> e <u>and</u> f <u>in a JW-algebra</u> A,

1) $RP(efe) = e - (e \cap (1-f))$.

2) $RP(fef) = f - ((1-e) \cap f)$.

PROOF. Replace f by $1-f$ in Lemma 11.

9. <u>Position p'</u>. Following Dixmier [3], we say that two projections e and f are in <u>position p'</u> if $e \cap (1-f) = 0 = (1-e) \cap f$. Projections e and f are said to be in <u>position p''</u> if $e \cap f = 0 = (1-e) \cap (1-f)$.

LEMMA 12. <u>Projections</u> e <u>and</u> f <u>are in position p'' if and only if they are complementary;</u> e <u>and</u> f <u>are in position p' if and only if</u> e <u>and</u> $1-f$ <u>are complements, or equivalently,</u> $1-e$ <u>and</u> f <u>are complements. For any two projections</u> e <u>and</u> f, <u>the projections</u> $(e \cup f) - f$ <u>and</u> $e - (e \cap f)$ <u>are in position p', as are the projections</u> $e - (e \cap (1-f))$ <u>and</u> $f - ((1-e) \cap f)$. <u>The latter are non-zero if and only if</u> $ef \neq 0$; <u>hence two nonorthogonal minimal projections in a JW-algebra are in position p'</u>.

Verification of these facts is left to the reader (see [21]).

PROPOSITION 9. <u>Let</u> e <u>and</u> f <u>be projections in position p''</u>. <u>If</u> s <u>is a symmetry exchanging</u> e <u>and</u> f <u>and if we set</u> $g = (1+s)/2$, <u>then</u> e <u>and</u> g <u>are in position p', as are</u> f <u>and</u> g.

PROOF. Let $x \in e \cap (1-g)$. Then $ex = x$ and $gx = 0$. Also $sx = 2gx - x = -x$ and since $fsx = sex = sx$ we have $x \in e \cap f = 0$. Hence $e \cap (1-g) = 0$. Next let $x \in (1-e) \cap g$. Then $gx = x$ and $ex = 0$. Now $sx = sgx = (2g-1)gx = gx = x$ so $fx = fsx = sex = 0$ and $x \in (1-e) \cap (1-f) = 0$. Thus $(1-e) \cap g = 0$. The proof that f and g are in position p' is similar.

The next result is a "Jordanized" version of a theorem of Takeda and Turumaru [21] (Theorem 1).

LEMMA 13. <u>Let</u> A <u>be a JW-algebra. Then for projections</u> $e, f \in A$ <u>the following are equivalent:</u>

1) e <u>and</u> f <u>are in position p'</u>.

2) e <u>and</u> f <u>are the range projections of</u> efe <u>and</u> fef <u>respectively</u>.

3) <u>For suitably chosen projections</u> $g, h \in A$, $e = (g \cup h) - h$ <u>and</u> $f = g - (g \cap h)$.

4) $\{efe\}^{\perp} = \{e\}^{\perp}$ and $\{fef\}^{\perp} = \{f\}^{\perp}$.

The proof of Lemma 13 is essentially the same as that of [21] (Theorem 1), with Lemma 11 replacing Lemma 5.3 of [10]. The equivalence of 2) and 4) results from the characterization of the range projection of a as the unique projection e satisfying $\{a\}^{\perp} = \{e\}^{\perp}$ (Proposition 4).

We come now to the main result of this section.

THEOREM 6. *Let e and f be projections in a JW-algebra A. If e and f are in position p' then there is a symmetry $s \in A$ exchanging e and f.*

PROOF. Let $a = e + f - 1$ and let $p = RP(a)$. Then if $a = s|a|$ is the canonical real polar decomposition, we have $s^2 = p \geq e \cup f$. As in the second half of the proof of Proposition 6, we have $s(efe)s = fef$ and $s(fef)s = efe$. Moreover, these relations are preserved if we replace the partial symmetry s by the symmetry $s + (1-p)$, so we shall assume that this has been done. Our task is to show $ses = f$, or what amounts to the same, $f = RP(ses)$. But since e and f are in position p', we have $\{efe\}^{\perp} = \{e\}^{\perp}$ and $\{fef\}^{\perp} = \{f\}^{\perp}$, by Lemma 13 (4). Now if $af = 0$, we also have $a(fef) = 0$ so $(sas)(efe) = (sas)s(fef)s = sa(fef)s = 0$ and $(sas)e = 0$. Finally $sa(ses) = 0$ and hence $a(ses) = 0$. Reversing these computations (using $s^2 = 1$), we see that $\{f\}^{\perp} = \{ses\}^{\perp}$. Hence $f = RP(ses) = ses$.

We remark that in Theorem 6 and in the second half of Proposition 6, it is somewhat neater to work with the operator $a = e + f - (e \cup f)$ rather than with $e + f - 1$. One then obtains $RP(a) = e \cup f$ and the extended symmetry required in each case is $s + ((1-e) \cap (1-f))$. Note also that if s is a symmetry with $ses = f$ (or a partial symmetry with $ses = f$ and $sfs = e$) then $a = s|a|$, where $a = e + f - 1$ (or $a = e + f - (e \cup f)$ if s is partial).

A number of important consequences of Theorem 6 are collected below.

COROLLARY 8. *For any two projections $e, f \in A$, there is a symmetry $s \in A$ with $s((e \cup f) - f)s = e - (e \cap f)$.*

PROOF. The projections $(e \cup f) - f$ and $e - (e \cap f)$ are in position p' by part 3) of Lemma 13.

COROLLARY 9. *Let A be a JW-algebra on whose projection lattice L a mapping $e \to d(e)$ is defined into some abelian group such that (1) $d(e \cup f) = d(e) + d(f)$ if $e \perp f$, (2) $d(ses) = d(e)$ for all symmetries $s \in A$, and (3) $d(e) = 0$ implies $e = 0$. Then L is modular and is therefore a continuous geometry.*

PROOF. Let e, f and g be projections in A with $e \leq g$. We consider the two expressions relevant for the modular law: $h = (e \cup f) \cap g$ and $k = e \cup (f \cap g)$. Now both h and k have the property that their union with f is $e \cup f$ and their intersection with f is $f \cap g$. Thus $h - (f \cap g)$ and $(e \cup f) - f$ are in position p' as are $(e \cup f) - f$ and $k - (f \cap g)$. By Theorem 6 and symmetry invariance (2), we have $d(h) = d(k)$. But $h \geq k$ in any lattice, so $d(h-k) = d(h) - d(k) = 0$ by (1), and by (3), $h = k$. Thus L is a modular lattice. A classical result of Kaplansky [12] shows that L, being a complete orthocomplemented modular lattice is a (reducible, perhaps) continuous geometry in the sense of von Neumann [17].

10. Relations with perspectivity. We recall that two projections are said to be *perspective* if they possess a common complement.

LEMMA 14. *If* $e, f \leq g$ *have a common relative complement* h *in* g, *then* $h \cup (1-g)$ *is a common complement for* e *and* f.

The proof can be found in [15] (Lemma 37, p. 23).

LEMMA 15. *Let* h *be a common complement for the projection* e *and* f. *If* g *is a projection which commutes with* h *and is orthogonal to* e *and* f, *then* $e \cup g$ *and* $f \cup g$ *are perspective with* $h \cap (1-g)$ *as a common complement.*

PROOF. Set $k = h \cap (1-g)$. The $(e \cup g) \cup k = e \cup (g \cup k) = 1$ since $g \cup k = g + (h - hg) = g \cup h$. Also $(e \cup g) \cap k = ((e \cup g) \cap (1-g)) \cap h = ((e+g)(1-g)) \cap h = e \cap h = 0$. Similarly, k is a complement for $f \cup g$.

THEOREM 7. *Let* e *and* f *be projections in a JW-algebra* A *which are exchanged by the symmetry* $s \in A$. *Then* e *and* f *are perspective in* A.

PROOF. Since s is the identity on $e \cap f$, we also have $s(e - (e \cap f))s = f - (e \cap f)$. Let $p = (e - (e \cap f)) \cup (f - (e \cap f))$. Then p commutes with s^+ and s^-. Let $s_0 = s_0^+ - s_0^-$, where $s_0^+ = ps^+$ and $s_0^- = ps^-$. Then s_0 is a symmetry in the JW-subalgebra pAp. Dropping down to pAp, we find that $e - (e \cap f)$ and $f - (e \cap f)$ are in position p" and are exchanged by the symmetry s_0. By Proposition 9, with $g = (p + s_0)/2$, we see that $e - (e \cap f)$ and g are in position p', as are $f - (e \cap f)$ and g (relative to p). By Lemma 12, $e - (e \cap f)$ and $f - (e \cap f)$ have $p - g$ as a common relative complement in p. Thus by Lemma 14, $e - (e \cap f)$ and $f - (e \cap f)$ have $1-g$ as a common complement. But $e \cap f$ is orthogonal to $e - (e \cap f)$ and

$f-(e\cap f)$ and commutes with $1-g$ (since $e\cap f$ commutes with $e-(e\cap f)$ and $f-(e\cap f)$, it commutes with their union). By Lemma 15, e and f are perspective, with $(1-g)\cap(1-(e\cap f))$ as a common complement.

COROLLARY 10. *If two projections e and f in a JW-algebra A are exchanged by a symmetry $s\in A$, and if d is any function defined on the projection lattice of A which identifies perspective projections, then* $d(e)=d(f)$.

THEOREM 8. *If e and f are perspective projections in a JW-algebra A, then there are symmetries $s, t\in A$ with $u^*eu=f$, where $u=st$.*

PROOF. If g is a common complement of e and f, then e and $1-g$ are in position p', as are f and $1-g$ (Lemma 12). Thus there are symmetries $s, t\in A$ with $ses=1-g$ and $tft=1-g$, by Theorem 6. Hence if $u=st$, then $u^*eu=f$.

PROPOSITION 10. *If e and f are orthogonal perspective projections in a JW-algebra A, then there is a symmetry $v\in A$ exchanging e and f.*

PROOF. By Theorem 8, there are symmetries $s, t\in A$ with $u^*eu=f$, where $u=st$. Now set $x=u^*e$. Then x is a partial isometry (not necessarily in $A+iA$) with $x^*x=e$ and $xx^*=f$. Using the relations $xe=x=fx$ and $x^2=0$, together with the orthogonality of e and f, we see that $v=x+x^*+(1-e-f)$ is a symmetry exchanging e and f. Since $x+x^*=est+tse=((e+t)s(e+t))-(ese+tst)\in A$ we have $v\in A$.

We call two projections e and f in A *equivalent*, written $e\sim f$, if there is a finite sequence $s_1, \dots, s_n$ of symmetries, each $s_i\in A$, with $u^*eu=f$, where $u=s_1\dots s_n$. It is easy to see that $\sim$ is an equivalence relation.

COROLLARY 11. $e\sim f$ *if and only if there are projections* $e=e_1, e_2, \dots, e_n=f$, *all in A with e_i and e_{i+1} perspective in A* $(i=1,\dots,n-1)$. *Thus equivalence coincides with projectivity.*

COROLLARY 12. *Let A be a JW-algebra with a modular projection lattice. Then $e\sim f$ if and only if e and f are perspective in A, and both $\sim$ and perspectivity are completely additive.*

PROOF. The transitivity and complete additivity of perspectivity in a continuous geometry (see Corollary 9) are classical results of von Neumann [17] and Kaplansky [12] respectively.

11. Equivalence and central covers. The relation of equivalence introduced in the last section is a special kind of unitary equivalence and is not even finitely additive in general. It is easy to see, however, that the projection lattice of a JW-algebra under our relation $\sim$ satisfies axioms (A), (B'), (D') and (M) for dimension lattices given by Loomis in [15] (Axiom (M) was verified in Theorem 4 and (D') is a consequence of Theorem 8 above. Axiom (A) is trivial and (B') is not difficult to check). Axiom (C) of Loomis fails in the worst possible way if the projection lattice is not modular. An extremely valuable special case of additivity holds however.

THEOREM 9. Let $\{e_i\}$ and $\{f_i\}$ be families of orthogonal projections in a JW-algebra A with $e = \mathrm{LUB}\, e_i$ and $f = \mathrm{LUB}\, f_i$. If e and f are orthogonal and if e_i and f_i are exchanged by a symmetry $s_i \in A$, then there is a symmetry $s \in A$ exchanging e and f.

PROOF. We have $s_i e_i s_i = f_i$. Let $x_i = s_i e_i$ so that $x_i^* x_i = e_i$ and $x_i x_i^* = f_i$. Set $g_i = (x_i + x_i^* + e_i + f_i)/2$. Then g_i is a projection in A. If $g = \mathrm{LUB}\, g_i$ and $s = 2g - 1$, then Kaplansky's proof of Lemma 3.1 in [10] applies without essential change showing that s is the required symmetry.

In case the families to be added are finite, the requirement that e and f be orthogonal can be relaxed somewhat.

PROPOSITION 11. Let $e = e_1 + e_2$ and $f = f_1 + f_2$ be projections in a JW-algebra A with $e_1 \perp e_2$ and $f_1 \perp f_2$. If $e_1 \perp f_2$ and $e_2 \perp f_1$ and if e_i and f_i $(i = 1, 2)$ are exchanged by a symmetry $s_i \in A$, then there is a symmetry $s \in A$ exchanging e and f.

PROOF. Let $p_i = e_i \cup f_i$ $(i = 1, 2)$. Then $p_1 \perp p_2$, since p_i is the range projection of $e_i + f_i$. Set $u = s_1 p_1$ and $v = s_2 p_2$. Then u and v are partial symmetries satisfying $u e_1 u = f_1$, $v e_2 v = f_2$ and $uv = 0$. From these relations we see that $s = u + v + (1 - p_1 - p_2)$ is the symmetry needed.

Two projections e and f in a JW-algebra A are related if there are projections $0 \neq e_1 \leq e$ and $0 \neq f_1 \leq f$ with $e_1 \sim f_1$. A projection e is termed invariant if e is not related to 1-e. We write $f \lesssim e$ if $f \sim e_1 \leq e$.

LEMMA 16. A projection e is invariant if and only if $f \lesssim e$ implies $f \leq e$.

The proof is exactly as in [15] (Lemma 21, p. 12) and is omitted.

PROPOSITION 12. <u>A projection in a JW-algebra is invariant if and only if it is central.</u>

PROOF. Let e be invariant in A. Then for any symmetry $s \in A$, $ses \sim e$ so $ses \leq e$ by Lemma 16 and $ses - e = s(e - ses)s \geq 0$. Thus $e = ses$ and $es = se$. Since A is uniformly generated by its symmetries (as well as by its projections), e is central.

If e is central and if $f \lesssim e$, then $u^*fu = e_1 \leq e$, where $u = s_1 \ldots s_n$ with each s_i a symmetry in A. Hence $e - f = u(e - e_1)u^* \geq 0$ and $f \leq e$. Thus e is invariant by Lemma 16.

COROLLARY 13. $C(e)$, <u>the central cover of</u> e, <u>is the smallest invariant projection</u> $\geq e$.

Note that e is invariant if and only if $sfs \leq e$ implies $f \leq e$ for any projection $f \in A$ and any symmetry $s \in A$; equivalently, $f \leq e$ implies $sfs \leq e$.

COROLLARY 14. $C(e) = \mathrm{LUB}\,\{f\colon\ f \lesssim e\}$. <u>Thus equivalent projections have the same central cover.</u>

PROOF. Set $p = \mathrm{LUB}\,\{f\colon\ f \lesssim e\}$. Clearly $f \leq C(e)$ if $f \lesssim e$, so $p \leq C(e)$. But p is invariant by [15] (Theorem 5, p. 15) and since $e \leq p$, we have $C(e) \leq p$ by Corollary 13, so $C(e) = p$.

COROLLARY 15. $C(e) \perp C(f)$ <u>if and only if</u> e <u>and</u> f <u>are unrelated</u>. <u>Thus a JW-algebra is a factor if and only if any two non-zero projections in it are related (ergodicity)</u>.

PROOF. If $C(e) \perp C(f)$, then $f \leq 1 - C(e)$. But $C(e)$ and $1 - C(e)$ are unrelated since $C(e)$ is invariant (Corollary 13). Thus e and f are unrelated.

If e and f are unrelated but $f\,C(e) = 0$, then f and $C(e)$ are related (in fact, by Lemma 12, f and $C(e)$ have non-zero subprojections in position p'). By [15] (Lemma 24, p. 15), f is related to some $g \lesssim e$, since $C(e)$ is the LUB of all $g \lesssim e$. Hence e and f are related, a contradiction. Thus $f\,C(e) = 0$ and $C(e) \perp C(f)$.

The following technique is useful in splitting off direct summands using certain families of projections.

PROPOSITION 13. <u>Let</u> P <u>be a set of projections in a JW-algebra</u> A, <u>such</u>

that $e \in P$ and $f \lesssim e$ imply $f \in P$. Set $p = \mathrm{LUB}\ P$. Then p is central and every projection in pAp can be written as the LUB of an orthogonal family of projections from P.

Loomis calls a family such as P "hereditary". His proof in [15] (Theorem 5, p. 15) applies and will not be repeated here.

12. Generalized comparability. The key to the results of this section is contained in

LEMMA 17. Let $e = e_o, e_1, \ldots, e_n = f$ be projections in a JW-algebra A with $e \perp f$ such that e_i and e_{i+1} are exchanged by a symmetry in A. Then e and f have non-zero subprojections which are exchanged by a symmetry in A.

PROOF. The proof is by induction on n. For $n = 1$, the conclusion is contained in the hypothesis. Suppose then that the conclusion holds for sequences of length n and let $e = e_o, e_i, \ldots, e_n, e_{n+1} = f$ be a sequence of the type under consideration. If $ee_n \neq 0$, then by Lemma 12, $g = e - (e \cap (1-e_n))$ and $h = e_n - ((1-e) \cap e_n)$ are non-zero subprojections of e and e_n, respectively, in position p'. By Theorem 6, g and h can be exchanged by a symmetry $s \in A$. If $t \in A$ is the symmetry exchanging e_n and e_{n+1}, let $k = tht$. Since $g \perp k$, we may, as in the proof of Proposition 10 construct the symmetry $v = (gst + tsg) + (1-g-k) \in A$ exchanging g and k.

If $ee_n = 0$, the induction hypothesis applies and yields subprojections $0 \neq g \leq e$ and $0 \neq h \leq e_n$ which are exchanged by a symmetry $s \in A$. If $t \in A$ is the symmetry exchanging e_n and e_{n+1}, set $k = tht$ and proceed as before.

COROLLARY 16. $C(e) = \mathrm{LUB}\{sfs: f \leq e\}$, where s ranges over all symmetries in A.

PROOF. Write $h = \mathrm{LUB}\ \{sfs: f \leq e\}$ and suppose $h \neq C(e)$. Then we can find projections $0 \neq f_1 \leq C(e) - h$ and $0 \neq e_1 \leq e$ with $e_i \sim f_1$. But this contradicts Lemma 17, since $e_1 \perp f_1$. Hence $h = C(e)$.

LEMMA 18. If e and f are orthogonal projections in a JW-algebra A, then we can write orthogonal sums $e = e_1 + e_2$ and $f = f_1 + f_2$, where e_1 and f_1 are exchanged by a symmetry in A and $C(e_2) \perp C(f_2)$.

PROOF. Let $\{e_i\}$ and $\{f_i\}$ be a maximal pair of families of orthogonal projections such that $e_i \leq e$, $f_i \leq f$ and e_i and f_i are exchanged by a symmetry

$s_i \in A$. Set $e_1 = \text{LUB } e_i$, $f_1 = \text{LUB } f_i$. By Theorem 9, e_1 and f_1 are exchanged by a symmetry $s \in A$ since $e \perp f$. Let $e_2 = e - e_1$, $f_2 = f - f_1$. If e_2 and f_2 (orthogonal) were related, then by Lemma 17 they would have non-zero subprojections which are exchanged by a symmetry in A. By Theorem 9 again, we could then add these symmetry exchanges to enlarge e_1 and f_1, contradicting maximality. Hence $C(e_2) \perp C(f_2)$ by Corollary 15.

We first dispose of the case of orthogonal projections.

LEMMA 19. *If e and f are orthogonal projections in a JW-algebra A, there is a central projection $h \in A$ such that eh and a subprojection of fh are exchanged by a symmetry $s \in A$; furthermore, $f(1-h)$ and a subprojection of $e(1-h)$ are exchanged by s.*

PROOF. Let $e = e_1 + e_2$ and $f = f_1 + f_2$ be the decompositions in Lemma 18, and set $h = C(f_2)$. Then $s e_1 s = f_1$ and since h is central, we have $eh = e_1h + e_2h = e_1h + e_2C(e_2)C(f_2) = e_1h$ and $s(eh)s = s(e_1h)s = f_1h \leq fh$. Also $f(1-h) = f_1(1-h) + f_2(1-h) = f_1(1-h) + f_2(1 - C(f_2)) = f_1(1-h)$ and $sf(1-h)s = s f_1(1-h)s = e_1(1-h) \leq e(1-h)$.

A strong form of generalized comparability is now within reach.

THEOREM 10 (THE COMPARISON THEOREM). *Given any two projections e and f in a JW-algebra A, there is a central projection $h \in A$ and a symmetry $s \in A$ with $s(eh) \leq fh$ and $s(f(1-h))s \leq e(1-h)$.*

PROOF. The case $e \perp f$ is covered by Lemma 19. Suppose then that $ef \neq 0$. By Lemma 12, the non-zero subprojections $e_1 = e - (e \cap (1-f))$ and $f_1 = f - ((1-e) \cap f)$ are in position p', and are exchanged by a symmetry $s_1 \in A$ by Theorem 6. Set $e_2 = e \cap (1-f)$ and $f_2 = (1-e) \cap f$. Since $e_2 \perp f_2$, Lemma 19 applies, giving a symmetry $s_2 \in A$ and a central projection $h \in A$ with $s_2(e_2h)s_2 \leq f_2h$ and $s_2 f_2(1-h) s_2 \leq e_2(1-h)$. But $e_1 \perp f_2$ and $e_2 \perp f_1$ so Proposition 11 enables us to piece these symmetry exchanges together. We now have a symmetry $s \in A$ satisfying $s(e_1 + e_2h)s \leq f_1 + f_2h$ and $s(f_1 + f_2(1-h))s \leq e_1 + e_2(1-h)$. On multiplying the first of these inequalities through by h, we obtain $s(eh)s = s(e_1h + e_2h)s \leq f_1h + f_2h = fh$. Multiplying the second inequality through by $1-h$ gives $s f(1-h)s \leq e(1-h)$.

COROLLARY 17. *Let A be a JW-factor and let e and f be any two projections in A. Then either $ses \leq f$ or $ses \geq f$, for some symmetry $s \in A$.*

COROLLARY 18. *If* e *and* f *are any two projections in a JW-algebra* A, *then we can write orthogonal sums* $e = e_1 + e_2$ *and* $f = f_1 + f_2$, *where* e_1 *and* f_1 *are exchanged by a symmetry in* A *and* $C(e_2) \perp C(f_2)$.

PROOF. In view of Lemma 18, we only need consider the case where $ef \neq 0$. Write $e_{11} = e - (e \cap (1-f))$, $e_{12} = e \cap (1-f)$, $f_{11} = f - ((1-e) \cap f)$ and $f_{12} = (1-e) \cap f$. Proceeding as in the proof of Theorem 10, we first obtain a symmetry $s_1 \in A$ exchanging e_{11} and f_{11}. Since $e_{12} \perp f_{12}$, there are, by Lemma 18, orthogonal decompositions $e_{12} = e_{13} + e_2$ and $f_{12} = f_{13} + f_2$ where e_{13} and f_{13} are exchanged by a symmetry $s_2 \in A$ and $C(e_2) \perp C(f_2)$. Since $e_{13} \perp f_{11}$ and $e_{11} \perp f_{13}$, Proposition 11 gives a symmetry $s \in A$ exchanging $e_1 = e_{11} + e_{13}$ and $f_1 = f_{11} + f_{13}$.

COROLLARY 19. *Given any two projections* e *and* f *in a JW-algebra* A, *there is a central projection* $h \in A$ *and a symmetry* $s \in A$ *with* $s(eh)s \leq fh$ *and* $s(1-e)(1-h)s \leq (1-f)(1-h)$.

PROOF. Take h and s as in Theorem 10. Then $s(eh)s \leq fh$ and $se(1-h)s \geq f(1-h)$. Thus $s(1-e)(1-h)s = (1-h) - se(1-h)s \leq (1-h) - f(1-h) = (1-f)(1-h)$.

13. Modularity and finiteness. In the theory of von Neumann algebras, it is customary to call a projection e "finite" if

(F) $f \sim e$ and $f \leq e$ imply $f = e$.

With the notion of equivalence introduced at the end of section 10, all central projections have this property. Since finite additivity (even for two summands) generally fails, a projection having property (F) may have subprojections which violate (F). Note also that $e \sim f$ if and only if $1-e \sim 1-f$; and if s is a symmetry with $f = ses$ and $f \leq e$, then $f = e$, since $f - e = s(e-f)s \geq 0$. Theorem 11 and Proposition 14 below clarify this situation. First, however, we need a lattice-theoretic result, which is undoubtedly well-known, but which is included for completeness.

Recall that an orthocomplemented lattice L is called *orthomodular* if $e \leq f$ implies $f = e \cup (f \cap (1-e))$, for all $e, f \in L$, where $1-e$ denotes the orthocomplement of e (axiom (M) of Loomis [15], p. 4).

LEMMA 20. *Let* L *be an orthomodular lattice. Then* L *is modular if and only if* $f \leq e$ *and* f *perspective with* e *imply* $f = e$, *for all* $e, f \in L$.

PROOF. Suppose L is not modular. Then L contains five elements e, f, g, h and k with $k<e<f<h$, $k<g<h$, $f \cap g = e \cap g = k$ and $f \cup g = e \cup g = h$. Let $p = g \cup (1-k)$. Then $e \cap p = 0 = f \cap p$ and by orthomodularity, $g = k \cup (g \cap (1-k)) = k \cup p$ since $k \leq g$. Thus $e \cup p = e \cup (k \cup p) = e \cup g = h$. Similarly, $f \cup g = h$. By Lemma 14, $p \cup (1-h)$ is a common complement for e and f.

Conversely, suppose e and f have a common complement g and that $f<e$. Then $\{0, 1, e, f, g\}$ is a non-modular sublattice and L is non-modular.

LEMMA 21. *If $\{e_i\}$ is a sequence of orthogonal projections in a JW-algebra A and if e_i and e_{i+1} are exchanged by a symmetry $s_i \in A$, then any e_i and e_k can be exchanged by a symmetry in A.*

PROOF. First we note that if $\{e_i\}$ is any (possibly uncountable) orthogonal family of projections such that each e_i is exchanged (by a symmetry in A) with a fixed projection e_1 in the family, then any e_i and e_k are exchanged by a symmetry in A. For if $se_is = e_1$ and $te_kt = e_1$, put $u = st$ and apply the proof of Proposition 10 to get a symmetry in A exchanging e_i and e_k. We proceed by induction. Suppose e_1 and e_n are exchanged by a symmetry $s \in A$. By assumption, there is a symmetry $t \in A$ exchanging e_n and e_{n+1}. By the remarks above, e_{n+1} and e_1 can be exchanged by a symmetry in A, as required.

For a projection $e \in A$, let $[0, e]$ be the set of all projections $f \in A$ with $f \leq e$. Note that $[0, e]$ is just the projection lattice of eAe.

We call a projection e *modular* if $[0, e]$ is a modular lattice. An important characterization of modular projections is given in

THEOREM 11. *For a projection e in a JW-algebra A, the following are equivalent:*

1) *e is a modular projection.*

2) *Every orthogonal family $\{e_i\}$ of projections, any two of which are exchanged by a symmetry in A, with each $e_i \leq e$, is necessarily finite.*

PROOF. 1) $\Rightarrow$ 2). Let $\{e_i\}$ be a family of the type described and let s_{ik} be a symmetry in A exchanging e_i and e_k. Since s_{ik} commutes with $e_i + e_k$ we can cut s_{ik} down to a partial symmetry $t_{ik} = s_{ik}(e_i + e_k)$ which still exchanges e_i and e_k. Moreover, $e\,t_{ik}\,e = t_{ik}$ so that $t_{ik} \in eAe$. We may then extend t_{ik} to a symmetry in eAe exchanging e_i and e_k there, using Proposition 6. By Theorem 7,

e_i and e_k are perspective in $[0, e]$. But $[0, e]$ is a continuous geometry (see the proof of Corollary 9), and a classical result of von Neumann [17] (Theorem 3.8, p. 21) forces $\{e_i\}$ to be finite.

2) $\Rightarrow$ 1). Suppose $[0, e]$ is not modular. By Lemma 20, we can find $0 \neq f \leq e$ and $g<f$ with g and f perspective in $[0, e]$ and hence in A by Lemma 14. Thus there are symmetries $s, t \in A$ with $ugu^* = f$, where $u = st$, by Theorem 8. Let $f_1 = u^*fu$ and $g_1 = f - f_1$ $(\neq 0)$. Repeating this process, we get $f_1 = f_2 + g_2$ where $f_2 = u^*f_1u$ and $g_2 = u^*g_1u$. Now $g_2 \leq f_1$, so $g_1 \perp g_2$. By the proof of Proposition 10, there is a symmetry in A exchanging g_1 and g_2. Continuing inductively, we obtain sequences $\{f_n\}$ and $\{g_n\}$ such that $f_n \perp g_n$, $f_n + g_n = f_{n-1}$, with g_n and g_{n+1} exchanged by a symmetry in A. Since the g_n's are orthogonal, Lemma 21 tells us that g_i and g_k are exchanged by a symmetry in A. Thus the sequence $\{g_n\}$ violates 2).

The preceeding theorem is a "relativized" result; the "absolute" version is given in

PROPOSITION 14. <u>For a JW-algebra</u> A, <u>the following are equivalent:</u>

1) <u>Every projection</u> $e \in A$ <u>has property</u> (F).

2) A <u>has a modular projection lattice.</u>

3) <u>Every orthogonal family of equivalent projections in</u> A <u>is finite</u>.

4) <u>Every orthogonal family of projections in</u> A, <u>any two of which are exchanged by a symmetry in</u> A, <u>is finite</u>.

PROOF. 1) $\Rightarrow$2). Let e, f and g be projections in A with $e \leq g$ and let $h = (e \cup f) \cap g$, $k = e \cup (f \cap g)$. As in the proof of Corollary 9, $k - (f \cap g) \sim h - (f \cap g)$ and $k \leq h$. By property (F), $h = k$.

The equivalence of 2) and 4) is a special case of Theorem 11.

2) $\Rightarrow$ 3). The projection lattice of A is a continuous geometry and $\sim$ is perspectivity (Corollaries 9 and 12). The result is immediate from 1) $\Rightarrow$ 2) of Theorem 11.

3) $\Rightarrow$ 4) is trivial.

4) $\Rightarrow$ 1). Again the projection lattice of A is a continuous geometry and $\sim$ reduces to perspectivity. Lemma 20 then yields 1).

Our next object will be to show that $e \cup f$ is modular if e and f are. Some preliminaries are needed.

LEMMA 22. *Let* $\{e_i\}$ *be a family of unrelated modular projections in a JW-algebra. If* $e = \text{LUB } e_i$, *then* e *is modular.*

PROOF. The $C(e_i)$'s are orthogonal and since the operation of taking central covers preserves arbitrary LUB's, the lattice $[0, C(e)]$ is the direct product of the $[0, C(e_i)]$. But $[0, e]$ is then the direct product of the modular lattices $[0, e_i]$ and is therefore modular.

COROLLARY 20. *The* LUB *of any family of central modular projections is modular.*

PROOF. From among the family of all central subprojections of the e_i's, choose a maximal orthogonal family $\{f_k\}$ and set $f = \text{LUB } f_k$, $\{e_i\}$ being the given family of central modular projections with LUB e. We must have $f = e$, for otherwise $e - f$ would be central and $(e - f)e_i \neq 0$ for some e_i, contrary to maximality. Since each f_k is central modular, e is too by Lemma 22.

The next lemma is a substitute for the fact that in a properly infinite von Neumann algebra, one can find a projection g with $1 \sim g \sim 1-g$. The relation $g \sim 1$ is of course impossible in our context unless $g = 1$.

LEMMA 23. *Let* A *be a JW-algebra and suppose that* 1 *is not modular. Then there is a projection* $g \in A$ *with the following properties:*

1) *There is an infinite orthogonal sequence* $\{g_i\}$ *of projections, any two of which are exchanged by a symmetry of* A, *with each* $g_i \leq g$ *and* $g = \text{LUB } g_i$.

2) *There is a symmetry* $r \in A$ *with* $rgr \leq 1-g$.

PROOF. By the proof of 2) $\Rightarrow$ 1) of Theorem 11, we can actually construct an infinite sequence $\{g_i\}_{i=1}^{\infty}$ of orthogonal projections, any two of which are exchanged by a symmetry in A. Define $g = \text{LUB}\{g_{2n} : n = 1, 2, \ldots\}$ and $h = \text{LUB}\{g_{2n-1} : n = 1, 2, \ldots\}$. Since $g \perp h$, we can find a symmetry $r \in A$ exchanging g and h by Theorem 9. Also $h \leq 1-g$ as required.

We are now able to show that the modular projections form a lattice.

THEOREM 12. *If* e *and* f *are modular projections in a JW-algebra* A, *then* $e \cup f$ *is modular.*

PROOF. Dropping down to $(e \cup f)A(e \cup f)$, we may assume $e \cup f = 1$. By Corollary 8, $(e \cup f) - f$ is modular, since it is exchanged with the modular projection

$e - (e \cap f)$ by a symmetry of A. Since $e \cup f = ((e \cup f) - f) + f$, we can assume that $ef = 0$.

Hence let e and f be orthogonal modular projections with $e + f = 1$, and suppose, to the contrary, that 1 is not modular. By Lemma 23, we can find a projection $g = \text{LUB } g_i$, where $\{g_i\}$ is an infinite orthogonal sequence and any two of the g_i's are exchanged by a symmetry in A; further, there is a symmetry $r \in A$ with $rgr \leq 1-g$. Now apply Corollary 19 to the pair g, e. There is a central projection $h \in A$ and a symmetry $s \in A$ with $s(gh)s \leq eh$ and $s(1-g)(1-h)s \leq (1-e)(1-h) = f(1-h)$. Hence gh and $(1-g)(1-h)$ are modular. But $g(1-h) \leq r(1-g)(1-h)r$ and the latter is modular. Thus both gh and $g(1-h)$ are modular, and by Lemma 22, g is modular, a contradiction, by Theorem 11. Finally, then, 1 is modular.

Theorem 12 has several important consequences.

COROLLARY 21. <u>In a JW-algebra</u> A, <u>if</u> e <u>is modular and</u> $f \sim e$, <u>then</u> e <u>and</u> f <u>are perspective</u>. <u>On the set of modular projections, perspectivity is transitive.</u> <u>If</u> e <u>is modular and</u> $f \sim e$ <u>with</u> $f \leq e$, <u>then</u> $f = e$ (<u>property (F)</u>). <u>Two equivalent modular projections in</u> A <u>can be exchanged by a symmetry in</u> A.

PROOF. Let e be modular and $s_1, \ldots, s_n$ symmetries in A with $u^*e u = f$, where $u = s_1 \ldots s_n$. Set $e_i = s_i \ldots s_1 e s_1 \ldots s_i$, where $i = 1, \ldots, n$ and let $e_o = e$ ($e_n = f$). Putting $p = \text{LUB}\{e_i : i = 0, 1, \ldots, n\}$, we see by Theorem 12 that pAp has a modular projection lattice. Now $e_{i+1} = s_{i+1} e_i s_{i+1}$ and since s_{i+1} commutes with $e_i + e_{i+1}$ we see that s_{i+1} also commutes with $p_i = e_i \cup e_{i+1} = RP(e_i + e_{i+1})$. Set $t_i = s_i p_i + (p - p_i)$ and let $w = t_1 \ldots t_n$. Then $w^* e w = f$, $w f w^* = e$ and each t_i is a symmetry in the JW-subalgebra pAp (see Proposition 6). By Theorem 7, we have $e = e_o, e_1, \ldots, e_n = f$ with e_i and e_{i+1} perspective in pAp. By Corollary 12, e and f are perspective in pAp, and hence in A, by Lemma 14.

Let e, f and g be modular projections in A with e perspective with f, and f perspective with g. By Theorem 8, $e \sim g$, so by the first part of the proof, e and g are perspective.

If e is modular and $f \sim e$ with $f \leq e$, then by the first part of the proof, e and f are perspective in some subalgebra pAp having a modular projection lattice. Hence $f = e$ by Lemma 20.

Finally, let e be modular and $e \sim f$. By Theorem 10, we can find a central projection $h \in A$ and a symmetry $s \in A$ with $s(eh)s \leq fh$ and $sf(1-h)s \leq e(1-h)$. But if $u^*e u = f$, u a finite product of symmetries from A, then $u^*(eh)u = fh$ and

$u^*e(1-h)u = f(1-h)$. Since fh and $e(1-h)$ are modular, $s(eh)s = fh$ and $sf(1-h)s = e(1-h)$ by property (F). Putting these together, we get $ses = f$.

The preceeding results lead us to the Schroeder-Bernstein Theorem for modular projections.

COROLLARY 22. *If e and f are modular projections in a JW-algebra A with $e \lesssim f$ and $f \lesssim e$, then e and f are exchanged by a symmetry in A.*

PROOF. By assumption, there are unitary operators u and v, each a finite product of symmetries from A, with $u^*eu = f_1 \leq f$ and $v^*fv = e_1 \leq e$. By Corollary 21, $v^*f_1v = e$. But $f_1 = vev^* \geq ve_1v^* = f$, so $f_1 = f$ and $e \sim f$. By Corollary 21 again, $ses = f$ for some symmetry $s \in A$.

14. The type decomposition. Let A be a JW-algebra and L its projection lattice. A subset $I \subset L$ is *join-dense* if for any $0 \neq e \in L$ there is an $f \in I$ with $0 \neq f \leq e$. By virtue of the completeness of L, an easy Zorn's Lemma argument shows that any projection $e \in L$ is the LUB of an orthogonal family of projections from I. An *ideal* in L is a subset $I \subset L$ such that i) $e, f \in I$ imply $e \cup f \in I$; and ii) $e \in I$ and $f \in L$ with $f \leq e$ imply $f \in I$. A *modular ideal* is an ideal, which as a sublattice of L, is a modular lattice. A *p-ideal* is an ideal $I \subset L$ such that $f \sim e$ and $e \in I$ imply $f \in I$.

COROLLARY 23. *In the projection lattice of a JW-algebra, the modular projections form a modular p-ideal.*

A JW-algebra A is *locally modular* if it has a faithful modular projection. Alternatively, A is locally modular if every direct summand contains a modular projection.

PROPOSITION 15. *If A is a locally modular JW-algebra then any non-zero projection contains a modular projection with the same central cover.*

PROOF. Given a non-zero projection $e \in A$. By assumption, there is a modular projection $p \in A$ with $C(p) = 1$. Since $e \neq 0$, $C(e)C(p) = C(e) \neq 0$, so e and p are related, by Corollary 15. Thus we can find $0 \neq f \leq e$ and $0 \neq g \leq p$ with $f \sim g$. But g is modular and hence f is also. Thus any non-zero projection contains a non-zero modular projection. Now let $\{e_i\}$ be a maximal family of modular projections such that $e_i \leq e$ and the $C(e_i)$ are orthogonal. Set $h = \text{LUB } e_i$. By Lemma 22, h is modular and clearly $C(h) \leq C(e)$. If $C(h) \neq C(e)$, then $e(C(e) - C(h))$

$\neq 0$ would contain a non-zero modular projection k. But then $C(k) \perp C(h)$ and $k \leq e$, contradicting maximality. Thus $C(h) = C(e)$.

PROPOSITION 16. The LUB of all modular projections in a JW-algebra A is a central projection e such that eAe is locally modular and $1-e$ contains no modular projections other than zero. This projection is the largest central projection f such that fAf is locally modular. The modular projections in eAe form a join-dense modular p-ideal in $[0, e]$.

PROOF. That e is central is clear from Corollary 23 and Proposition 13, as is the last statement.

Let $\{p_i\}$ be a maximal family of unrelated modular projections in eAe and set $p = \text{LUB } p_i$. By Lemma 22, p is modular. If $C(p) \neq e$, then $(e - C(p)) \cap e = e - C(p) \neq 0$ so that $f = (e - C(p)) \cap g \neq 0$, for some modular projection g. But then we have $C(f) \leq e - C(p)$, contradicting maximality. Thus $C(p) = e$ and eAe is locally modular. The rest is clear.

A JW-algebra A with a modular projection lattice will be called modular. We say that A is properly non-modular if A has no central modular projections except zero. And A is purely non-modular if A contains no modular projections except zero.

PROPOSITION 17. Let A be a JW-algebra. There is a largest central projection e (resp. f, g, h) such that eAe (resp. fAf, gAg, hAh) is modular (resp. locally modular, properly non-modular, purely non-modular). These four projections are related by

$$eg = 0, \quad e + g = 1; \quad fh = 0, \quad f + h = 1; \quad e \leq f \text{ and } h \leq g.$$

PROOF. Let e be the LUB of all central modular projections in A and set $g = 1-e$. Corollary 20 gives the required information about e and g. Next take f to be the LUB of all modular projections in A and set $h = 1-f$. Proposition 16 shows that f and h are the required projections and clearly $e \leq f$ and $h \leq g$.

THEOREM 13. Any JW-algebra decomposes uniquely into five summands as follows:

1) Type I modular.
2) Type I properly non-modular, locally modular.
3) Type II modular.

4) Type II properly non-modular, locally modular.

5) Type III purely non-modular.

A JW-factor has one and only one of these five types.

PROOF. First we split off the type I portion of A as in §7. Since each abelian projection is manifestly modular, the type I summand is locally modular, and can be further split into a modular summand and a properly non-modular summand. The non-type I part is then split into three summands in accordance with Proposition 17. The last statement is clear.

One virtue of our type decomposition is that it coincides with the classical one if our JW-algebra A happens to be the s.a. part of a von Neumann algebra, even though the equivalence relations differ in the "infinite" cases. It is clear that the two notions of type I agree in A if $A + iA$ is a von Neumann algebra.

If e is a finite projection, then it is well-known [10] (Theorem 6.3) that $[0, e]$ is a modular lattice so that eAe is a modular JW-algebra. Now for any two orthogonal equivalent (in the sense of von Neumann) projections e and f in A, let $x \in A + iA$ be the partial isometry implementing the equivalence. Thus $e = x^*x$ and $f = xx^*$ and it is easily seen that $s = x + x^* + (1-e-f)$ is a symmetry in A exchanging e and f. Now a projection e is finite if and only if every orthogonal family $\{e_i\}$ of (von N) equivalent projections, with each $e_i \leq e$, is finite. By the remarks above, together with Theorem 11, modular projections are finite. Thus the two decompositions agree.

It has recently been shown by Peter Fillmore that unitary equivalence coincides with perspectivity (so that the latter is transitive) in any von Neumann algebra. Now in the finite (= modular) portion, we know that equivalence ($\sim$) in the present sense agrees with perspectivity and symmetry exchange (Corollaries 12 and 21). But in the properly infinite (= properly non-modular) portion, our equivalence relation ($\sim$) is really projectivity (Corollary 11), which by Fillmore's result reduces to perspectivity, and hence to unitary equivalence.

15. Some structure theory. With the Comparison Theorem (Theorem 10) at hand, we can easily derive a wealth of structural information about JW-algebras.

We denote by $eAf + fAe$ the set of $eaf + fae$, where $a \in A$.

LEMMA 24. Let e and f be projections in a JW-algebra A. Then $eAf + fAe = 0$ if and only if $C(e) \perp C(f)$.

PROOF. If $C(e)C(f) = 0$, then $eaf = eC(e)aC(f)f = 0$. Conversely, sup-

pose $eAf + fAe = 0$, but $C(e)C(f) \neq 0$. Then e and f have subprojections $0 \neq e_1 \leq e$ and $0 \neq f_1 \leq f$ which are exchanged by a symmetry $s \in A$, $s e_1 s = f_1$ (Corollary 18). Now $f_1 = se_1sf_1 = 0$ (since $e_1sf_1 = e_1(esf)f_1$ and $esf = 0$), a contradiction. Thus $C(e) \perp C(f)$.

We now arrive at the fundamental relation between a JW-algebra A and its center Z.

THEOREM 14. For any projection $e \in A$, the center of eAe is Ze.

PROOF. Let f be a central projection in eAe. Since $f \leq e$, $e - f$ is a projection and $fa(e-f) = f(eae)(e-f) = (eae)f(e-f) = 0$. Thus $C(f) \perp C(e-f)$ by Lemma 24, so $C(f)(e-f) = 0$ and $f = C(f)e \in Ze$.

For an abelian projection e, we have $eAe = Ze$.

COROLLARY 24. If A is a JW-factor and e is any projection in A, then eAe is a JW-factor.

COROLLARY 25. In any JW-factor, every abelian projection is minimal. If a JW-algebra has a faithful minimal projection, it is a type I factor.

PROOF. The first statement if clear. If e is a faithful minimal projection in A, then eAe is one-dimensional and for any central projection g, we have $ge = e$ or $ge = 0$. For $ge = e$, $C(e) \leq g$ and $g = 1$. If $ge = 0$, $(1-g)e = e$ and $C(e) \leq 1-g$ so $g = 0$. Thus A is a factor (of type I).

PROPOSITION 18. For a projection e in a JW-algebra A, the following are equivalent:

1) e is abelian.

2) For any projection $f \in A$ with $f \leq e$, f and $e-f$ are unrelated (e is "simple" in the terminology of [15]).

3) For any projections $f, g \in A$ with $f, g \leq e$, we have $C(fg) = C(f)C(g)$, and $C(f) \leq C(g)$ implies $f \leq g$.

PROOF. 1) $\Rightarrow$ 2). If $f \leq e$, then $f = C(f)e$ by the proof of Theorem 14. Similarly $e-f = C(e-f)e$, so $C(f)C(e-f)e = 0$ and hence $C(f)C(e-f)C(e) = 0$. But $C(e) = C(f) \cup C(e-f)$, so $C(f)C(e-f) = 0$ and by Corollary 15, f and $e-f$ are unrelated.

2) $\Rightarrow$ 1) If $f \leq e$, then $C(f) \perp C(e-f)$, since f and $e-f$ are unrelated (Corollary 15). Hence $f = C(f)e$ and $eAe = Ze$ is abelian as is e.

2) $\Rightarrow$ 3). By 2) $\Rightarrow$ 1), $f = C(f)e$ and $g = C(g)e$. Since $C(f), C(g) \leq C(e)$

we have $C(f)C(g)C(e) = C(f)C(g)$. But $fg = C(f)C(g)e$, so $C(fg) = C(f)C(g)C(e)$. The other part of 3) is clear.

3) $\Rightarrow$ 2) is obvious.

We proceed to complete our list of abelian properties.

LEMMA 25. *Let* $\{e_i\}$ *be a family of unrelated abelian projections in a JW-algebra* A *and set* $e = \text{LUB}\, e_i$. *Then* e *is abelian.*

PROOF. The proof is the same as that of Lemma 22, with "modular" replaced by "abelian" (= distributive).

PROPOSITION 19. *In any JW-algebra, the* LUB *of all abelian projections is a central projection (the one described in Theorem 5). In a type I JW-algebra, the abelian projections generate a join-dense modular p-ideal in the projection lattice; and any projection contains an abelian projection with the same central cover.*

The proof is virtually the same as Propositions 15 and 16 and is omitted. The key facts needed are found in Proposition 13 and Theorem 12.

LEMMA 26. *Let* e *and* f *be abelian projections in a JW-algebra* A, *with* $C(e) = C(f)$. *Then* e *and* f *are exchanged by a symmetry in* A.

PROOF. By Corollary 18, we can write orthogonal sums $e = e_1 + e_2$ and $f = f_1 + f_2$, where $se_1s = f_1$ for some symmetry $s \in A$ and $C(e_2) \perp C(f_2)$. Since the operation of taking central covers preserves LUB's, we have $C(e) = C(e_1) + C(e_2)$ and $C(f) = C(f_1) + C(f_2)$ by Proposition 18 (3). By assumption, $C(e) = C(f)$ and since equivalent projections have the same central cover (Corollary 14), $C(e_1) = C(f_1)$. Thus $C(e_2) = C(f_2) = 0$ and $e_2 = f_2 = 0$ so that $ses = f$.

COROLLARY 26. *Any two minimal (=abelian) projections in a (type I) JW-factor* A *can be exchanged by a symmetry in* A.

A JW-algebra A is said to be *homogeneous* if there is an orthogonal family $\{e_i\}$ of abelian projections, any two of which are exchanged by a symmetry in A, such that $\text{LUB}\, e_i = 1$. A homogeneous algebra is evidently type I. From Proposition 13 and Corollary 26, we see that any JW-factor of type I is homogeneous.

Before proceeding, we note that the "spectral multiplicity" of a homogeneous JW-algebra is unique.

THEOREM 15. *Let* Z *be a commutative JW-algebra. Then there is an orthogonal family* $\{h_i\}$ *of projections in* Z *with* LUB 1, *such that each* Zh_i *is countably decomposable. Further, if* A *is any homogeneous JW-algebra, then the cardinal numbers of any two homogeneous partitions of the identity are the same.*

PROOF. Our algebra Z possesses a separating family of vector states and these are completely additive on projections. The support h_1 of such a state is a non-zero projection in Z and the state becomes faithful on Zh_1. Countable decomposability of Zh_1 is then immediate. An easy application of Zorn's Lemma proves the first statement.

The proof of the second statement is identical with the argument given by Kaplansky in [11] (Theorem 4, p. 471), the convergence arguments for "infinite sums" being considerably simpler, since the strong and weak operator topologies are at our disposal.

We call a JW-algebra A atomic if every projection contains a minimal projection (i.e. an atom). An atomic algebra in which any two orthogonal atoms are related (and hence equivalent) is easily seen to be a type I factor.

PROPOSITION 20. *Let A be a JW-algebra, Z its center and p an atom in A. Then $h = C(p)$ is an atom in Z and hAh is a type I factor. The LUB of all atoms is a central projection e such eAe is decomposed uniquely into the direct product of type I factors and $1-e$ contains no atoms other than zero. This projection is the largest central projection f such that fAf is atomic.*

PROOF. If p is an atom, pAp is one-dimensional and $pAp = Zp$. Since $zh \to zp$ is an isomorphism of Zh with Zp, h is an atom in Z.

Let $\{e_i\}$ be the family of all atoms and set $e = \text{LUB } e_i$. Proposition 13 implies that e is central and that eAe is atomic. By the minimality of the $C(e_i)$'s in Z and Lemma 26, e_i and e_k are either unrelated or exchanged by a symmetry in eAe. If $h_i = C(e_i)$ (neglecting multiplicities) then $h_iA\,h_i$ is a type I factor (Corollary 25) and $e = C(e) = \text{LUB } C(e_i) = \text{LUB } h_i$ gives the desired decomposition.

We now obtain the general type I decomposition.

THEOREM 16. *Let A be a type I JW-algebra. Then there is a family $\{h_i\}$ of orthogonal central projections with $\text{LUB } h_i = 1$, such that each $h_iA\,h_i$ is homogeneous.*

PROOF. Let $\{e_i\}$ be a maximal family of orthogonal faithful abelian projections and set $e = \text{LUB } e_i$. Now $1-e$ is not faithful, since otherwise it would contain a faithful abelian projection by Proposition 19, contrary to maximality. Set $h = 1-C(1-e)$. Then $(1-e)h = 0$ and $h = he \neq 0$. Since h commutes with each e_i, $h = he = \text{LUB } he_i$. By Lemma 26, any two of the e_i's are exchanged by a symmetry in A, and the same symmetries exchange the he_i's. Since each he_i is abelian, hAh is homogeneous.

We have just shown that any JW-algebra of the type under scrutiny contains a non-trivial homogeneous direct summand. A simple Zorn's Lemma argument completes the proof.

On the "continuous" side of the picture we have

THEOREM 17. Let A be a JW-algebra with no type I portion. Then any projection in A can be split into two orthogonal halves which are exchanged by a symmetry in A.

PROOF. Given e, let $(\{f_i\}, \{g_i\})$ be a maximal pair of families of orthogonal projections such that $f_i, g_i \leq e$, $f_i \perp g_i$ and f_i and g_i are exchanged by a symmetry $s_i \in A$. Set $f = \text{LUB } f_i$ and $g = \text{LUB } g_i$. We have $f \perp g$ and Theorem 9 furnishes a symmetry $s \in A$ exchanging f and g. Our task is to show that $e = f + g$, and for this it is enough to show that if e is not abelian, then we can find $0 \neq e_1 \leq e$ and $0 \neq e_2 \leq e$ with $e_1 \perp e_2$, e_1 and e_2 exchanged by a symmetry in A. But if e is not abelian, there is a non-central projection $f \in eAe$. If $C(f)C(e-f) = 0$ then $f = C(f)e$ is central in eAe, a contradiction. Thus f and $e-f$ are related, so by Corollary 18, we can find $0 \neq e_1 \leq f$ and $0 \neq e_2 \leq e-f$, e_1 and e_2 exchanged by a symmetry in A.

An abelian projection can never be split into two orthogonal equivalent pieces (Proposition 18 (2)).

We close this section with some results on "strong semisimplicity" and "weak centrality".

LEMMA 27. If A is a modular JW-algebra, and if I is a non-zero Jordan ideal in A, then I contains a non-zero central projection.

PROOF. Let $0 \neq a \in I$. By the Spectral Theorem (see [10], Lemma 2.1), we can find a non-zero projection $e \in A$ and an operator $b \in A$ with $e = ab = ba$, so that $e \in I$. Next let $\{e_i\}$ be a maximal family of orthogonal projections, any two of which are exchanged by a symmetry in A, with $e_1 = e$. This family is necessarily finite by Proposition 14 (4) and we list the projections: $e = e_1, e_2, \ldots, e_n$. Set $f = 1 - \sum_{i=1}^{n} e_i$ and apply Generalized Comparability (Theorem 10) to f and e_1; there is a central projection $h \in A$ and a symmetry $s \in A$ with $s(fh)s \leq e_1h$ and $s\,e_1(1-h)\,s \leq f(1-h)$. Now $e_1h \neq 0$, for if $e_1h = 0$, then $se_1s = se_1(1-h)s \leq f(1-h) = f$, contrary to maximality. Also $h \neq 0$ and $h = fh + (1-f)h = fh + \sum_{i=1}^{n} e_ih$ with $s(fh)s \leq e_1$ and $s_i(e_ih)s_i \leq e_1$ for suitable symmetries $s_i \in A$. To show $h \in I$ it is enough to show that if e and f are projections and s is a symmetry, all

in A, with $ses \leq f \in I$, then $e \in I$. But $ses = (ses)f \in I$ and $e = s(ses)s = 2(s \circ (s \circ (ses))) - ses \in I$.

We say that a JW-algebra A is simple if it has no non-trivial Jordan ideals. A is strongly semisimple if the intersection of its maximal Jordan ideals is zero [22].

COROLLARY 27. *Any modular JW-factor is simple.*

THEOREM 18. *Any modular JW-algebra A is strongly semisimple.*

PROOF. Let $I = \bigcap M$, where M runs over the maximal Jordan ideals of A. If $I \neq 0$, I contains a non-zero central projection h. Since $1-h \neq 1$ and $(1-h)A(1-h)$ is a (proper) Jordan ideal (Proposition 5), it can be extended, via Zorn, to a maximal proper Jordan ideal M. By the definition of I, $h \in M$. But $1-h \in M$ by construction, so $1 \in M$, a contradiction. Thus $I = 0$.

We pause to note several facts about the projections in a Jordan ideal.

LEMMA 28. *Let A be a JW-algebra, L its projection lattice. If I is a Jordan ideal in A, then $P = I \cap L$ is a p-ideal in L. Moreover, if I is uniformly closed, it is the smallest uniformly closed Jordan ideal containing P.*

PROOF. If $e \in P$ and $e \sim f$ let $u = s_1 \ldots s_n$, each s_i a symmetry in A, with $u^*eu = f$. Now $s_1 e s_1 = 2(s_1 \circ (s_1 \circ e)) - e \in I$ and hence $f = u^*eu \in P$ by induction. If $e \in P$ and $f \leq e$ then $f = f \circ e$, so $f \in P$. For $e, f \in P$ we have $(e \cup f) - f \sim e - (e \cap f) \leq e$ and $(e \cup f) - f \in P$. Hence $e \cup f \in P$ and P is a p-ideal.

Finally, suppose that P generates the Jordan ideal J. Clearly $P \subset J \cap L$. But $I \supset J$, so $I \cap L \supset J \cap L$ and hence $P = J \cap L$. Now take any $a \in I$ and any $\epsilon > 0$. By [10] (Lemma 2.1) we can find a non-zero projection $e \in A$ and $b \in A$ with $e = ab = ba$, $ea = ae$ and $\| a - ea \| < \epsilon$. Thus $e \in P$ and $ea \in J$ so that $a \in J^-$ and $I = J^-$.

LEMMA 29. *Let P be a p-ideal in L and let I be the uniform closure of $\{a \in A: RP(a) \in P\}$. Then I is a Jordan ideal, the smallest closed one containing P, and $I \cap L = P$.*

PROOF. Set $K = \{a \in A: RP(a) \in P\}$. We show first that K is an absolute order ideal. Since $RP(|a|) = RP(a)$ we have $|a| \in K$ if $a \in K$. Also $RP(a+b) \leq RP(a) \cup RP(b)$ so $a + b \in K$ whenever $a, b \in K$. Thus K is a linear subspace closed under $a \to |a|$. Now take $0 \leq a \leq b \leq 1$ (normalizing) with $b \in K$ and let $e = RP(b)$. Then $0 \leq a \leq b \leq e$ and therefore $ae = a = ea$. This shows $RP(a) \in P$ since $RP(a) \leq e \in P$ and $a \in K$ so that K is indeed an absolute order ideal, as is its uniform closure I.

Next we show that for e a projection in A and $e \in K^+$, $eae \in K$. Let $p = RP(a)$ and normalize so that $0 \leq a \leq 1$ and hence $0 \leq a \leq p$. Then $0 \leq eae \leq epe$, so it is enough to show $epe \in K$. Now $s = 2e - 1$ is a symmetry, so $sps \in P$ and $epe + (1-e)p(1-e) = (p + sps)/2 \in K$. But $0 \leq epe \leq epe + (1-e)p(1-e)$, so $epe \in K$ and $eae \in K$ as claimed.

For $a \in K$ arbitrary, $a = a^+ - a^-$ with $a^+, a^- \in K$ and $eae \in K$ for any projection $e \in A$. Moreover, this property passes to the uniform closure I of K; thus $e \in A$ and $a \in I$ imply $eae \in I$.

But any symmetry $s \in A$ can be written as $s = 2e - 1$, where e is a projection. Hence for $a \in I$, $(a + sas)/2 = eae + (1-e)a(1-e) \in I$ and therefore $sas \in I$. Also $(1+s)a(1+s) = 4\, eae \in I$ so that $2\,(s \circ a) = sa + as = (1+s)a(1+s) - (a + sas) \in I$. Finally then $s \circ a \in I$. But A is uniformly generated by its symmetries, and since I is uniformly closed, it is a Jordan ideal.

If J is any closed Jordan ideal containing P, then if $e = RP(a) \in P$ we have $e \in J$ and $a = a \circ e \in J$ so $I \subset J$.

For the last statement, let e be a projection in I, $\|e - a_n\| \to 0$ with each $a_n \in K$. Set $e_n = RP(a_n) \in P$ and observe that e_n approaches a_n as a_n approaches e. Specifically, we can choose n large enough so that $\|e - a_n\| < 1/2$ and $\|a_n - e_n\| < 1/2$. Then $\|e - e_n\| = \|(e-a_n) + (a_n-e_n)\| \leq \|e - a_n\| + \|a_n - e_n\| < 1/2 + 1/2 = 1$. By Proposition 7, e and e_n are exchanged by a symmetry in A, so $e \sim e_n \in P$ and $e \in P$. Thus I contains no new projections.

THEOREM 19. <u>There is a one-to-one correspondence between the uniformly closed Jordan ideals of a JW-algebra</u> A <u>and the p-ideals of its projection lattice</u> L. <u>The correspondence is given by</u> $I \to I \cap L$.

PROOF. This is immediate from Lemmas 28 and 29.

We can now show that JW-algebras are "weakly central" [22].

THEOREM 20. <u>Let</u> A <u>be any JW-algebra</u>, Z <u>its center, and let</u> M_1, M_2 <u>be any two maximal Jordan ideals in</u> A. <u>Then</u> $M_1 \cap Z \subset M_2 \cap Z$ <u>implies</u> $M_1 = M_2$.

PROOF. Suppose $M_1 \neq M_2$ and let e be a projection in M_1, f a projection in M_2 not in M_1. By Theorem 10 we can find a central projection $h \in A$ and a symmetry $s \in A$ with $s(eh)s \leq fh$ and $e(1-h) \geq s\,f(1-h)s$. Since $e(1-h) \in M_1$, we have $f(1-h) \in M_1$. Hence $fh \notin M_1$, for otherwise we would have $f \in M_1$. Thus $h \in M_1$. Now A/M_1 is a simple (abstract) Jordan algebra and if $1-h \notin M_1$,

h would map to a central idempotent in A/M_1 and give rise to a proper Jordan ideal there. Therefore $1-h \in M_1$ and $1-h \in M_2$ by the assumption $M_1 \cap Z \subset M_2 \cap Z$. Also $e(1-h) \in M_2$ and since $f \in M_2$ we have $fh \in M_2$ as well as $eh \in M_2$. But then $e = eh + e(1-h) \in M_2$. Thus $M_1 \cap L \subset M_2 \cap L$ and by Theorem 19, $M_1 \subset M_2$. Hence $M_1 = M_2$ as claimed.

THEOREM 21. <u>There is a one-to-one correspondence between the space of maximal Jordan ideals in a JW-algebra A and the space of maximal ideals of its center Z. Furthermore, if the space for A is given the Stone topology and if the space for Z is given the Gelfand topology, this correspondence is a homeomorphism.</u>

PROOF. First observe that any proper closed ideal N of Z generates a proper Jordan ideal in A. For if P is the set of projections in L which are bounded above in $N \cap L$, then P is a p-ideal in L, and so generates a closed Jordan ideal I in A with $I \cap L = P$. If $I = A$, then $1 \in P$ and $1 \in N$, a contradiction. Hence I is a proper closed Jordan ideal containing N.

Next suppose that $M \cap Z$ is contained in the maximal Jordan ideal M', where M is a maximal Jordan ideal. By Theorem 20, $M = M'$, so that $M \cap Z$ is contained in exactly one maximal Jordan ideal, namely M. By the opening remark of the proof, if N is a maximal ideal in Z containing $M \cap Z$, we have $N \subset M$ so that $N = M \cap Z$. The same remark shows that every maximal ideal in Z has this form. By Theorem 20, the mapping $M \to M \cap Z$ is one-to-one and it is easily seen to be a homeomorphism.

16. <u>Dimension functions</u>. Some recent results of Arlan Ramsay [18] can be applied to our situation to construct dimension functions. Ramsay's principal result is this: A complete orthomodular lattice L can be furnished with an equivalence relation making it into a dimension lattice with no type III part if and only if L has a join-dense modular ideal. Given such a lattice, Ramsay gets the equivalence relation by constructing a dimension function d on L having the following properties:

d 1) The values of d are continuous extended real-valued functions on the Stone space of the center of L.

d 2) $d(e) = 0$ if and only if $e = 0$.

d 3) d is completely additive, i.e., if $\{e_i\}$ is an orthogonal family in L with $e = \text{LUB } e_i$, then $d(e) = \text{LUB } d(e_i)$.

d 4) If e and f are perspective in L, then $d(e) = d(f)$.

d 5) The principal ideal $[0, e]$ is modular if and only if $d(e)$ is finite except on a nowhere dense set.

Recall that the center of L is the set of all elements with unique complements. Ramsay's construction insures that every central element is invariant (an invariant element always has a unique complement [15] (proof of Theorem 2)).

LEMMA 30. A projection in a JW-algebra is central if and only if it has a unique complement.

PROOF. If e is central, then for any projection f we have $e \cap f = ef$, $e \cup f = e + f - ef$. Hence if f is a complement of e, $ef = 0$ and $f = 1-e$.

Conversely, suppose $1-e$ is the unique complement of e. If e is not central, $C(e)C(1-e) \neq 0$ by Corollary 15 so by Corollary 18 we can find subprojections $0 \neq g \leq e$ and $0 \neq h \leq 1-e$ with $sgs = h$ for some symmetry $s \in A$. Let $t = s(g+h)$ and set $k = (g + h - t)/2$; k is a projection with $k(1-e-h) = 0$. Setting $f = (1-e)-h+k$, we obtain a projection, which we claim is a complement of e. (1) For $m = e \cup f$, $m \geq e, k$ and so $mk = k$ and $me = e$. Also $mg = g$ and $mf = f$, implying $mh = h$ and $m \geq h$. But then $m \geq e + h + (f-k) = 1$. (2) Let $n = e \cap f$. From $en = n$ and $(f-k)e = 0$ we get $(f-k)n = 0$. Thus $n = fn = kn$. Also $2\,eke = g$ so that $2n = 2ekn = gn \leq n$ and $n = 0$. Hence f is a complement of e, so that $h = k$ and $g = h = 0$, a contradiction. Finally, then, e must be central.

In a locally modular JW-algebra, the modular projections form a join-dense modular p-ideal (Proposition 16) and we have

THEOREM 22. The projection lattice of any locally modular JW-algebra has a symmetry invariant dimension function d satisfying conditions d 1) - d 5) above.

PROOF. Symmetry invariance of d follows from d 4) and Corollary 10. The existence of d is one of Ramsay's results [18] cited above.

17. Dimension lattice structure. Despite the richness of our dimension theory, it is still desirable to have complete additivity of equivalence in the non-modular cases. We shall now remove this apparent flaw by coarsening our equivalence relation until it becomes completely additive, being careful not to disturb the type decomposition.

Because the modular summand of a JW-algebra has a natural and eminently satisfactory dimension lattice structure, we shall tacitly assume in this section that

this summand has been removed so that our algebra is properly non-modular. It will also be convenient to treat the locally modular and purely non-modular cases separately.

Assume first that A is a locally modular JW-algebra with projection lattice L, and let d be Ramsay's dimension function described in Theorem 22. We define $e \approx f$ to mean $d(e) = d(f)$. In [18] (Theorem 4.23) Ramsay proves the following result in a more general setting.

THEOREM 23. $(L, \approx)$ <u>is a dimension lattice satisfying axioms</u> (A), (B), (C), (D') <u>and</u> (M) <u>of Loomis</u> [15] <u>such that central projections are</u> $\approx$<u>-invariant and modular projections are</u> $\approx$<u>-finite</u>.

It is natural to inquire whether Theorem 23 gives the only possible dimension lattice structure on the projections of a locally modular JW-algebra, in such a way that $\approx$ agrees with $\sim$ (= perspectivity, by Corollary 21) on the p-ideal of modular projections. Under a rather reasonable countability assumption, the answer to our inquiry is affirmative. We are indebted to Arlan Ramsay for pointing this out and for supplying the uniqueness proof sketched below. The countability assumption is:

(*) Each orthogonal family of modular projections having a common central cover is countable.

Any JW-algebra acting on a separable Hilbert space is countably decomposable and therefore satisfies (*). A locally modular JW-factor satisfies (*) if and only if it is countably decomposable. It is not hard to show that any direct product of JW-algebras satisfying (*) also satisfies (*).

Assume then that A is locally modular and satisfies (*), and let $\approx_1$ and $\approx_2$ be two equivalence relations on L satisfying axioms (A), (B), (C) and (D') of Loomis [15], each agreeing with $\sim$ on modular projections.

For e and f in L with $e \approx_1 f$, the largest central projection h with he modular is also the largest central projection with hf modular, and $he \approx_2 hf$. Thus we may assume that eAe and fAf are properly non-modular. Using (*) and an argument much like the one given in Theorem 16, we can partition e into an infinite orthogonal sequence of faithful (in eAe) modular projections (first a direct product is obtained in which each summand has this form, then a sequence is pieced together across the summands). By induction, using axiom (B) of [15], we can transfer this sequence under e to a similar sequence under f. This shows that f dominates

e in the $\approx_2$ relation. By symmetry $e \approx_2 f$, and on reversing the roles of $\approx_1$ and $\approx_2$, we obtain uniqueness.

A simple example serves to point out the technical difficulties which arise in the absence of some countability restriction. Let A be the JW-algebra of all s.a. operators on an inseparable Hilbert space. One dimension lattice structure is gotten from equality of Hilbert space dimension. A different structure is obtained by ignoring the distinction between infinite cardinals.

We are now left with the question of what to do about the type III purely non-modular portion of the general JW-algebra. A general result of Ramsay [18] (Theorem 7.1) furnishes a simple answer.

THEOREM 24. *Let* A *be a JW-algebra of type III with projection lattice* L. *For* $e, f \in L$ *define* $e \approx f$ *to mean that* e *and* f *have the same central cover. Then* $(L, \approx)$ *is a type III dimension lattice satisfying axioms* (A), (B), (C), (D') *and* (M) *of Loomis* [15] *such that central projections are* $\approx$*-invariant*.

18. *The trace.* Let A be a JW-algebra and let G be the set of all finite products of symmetries from A. Then G is a group of unitary operators, acting on A as a group of linear order isometries by the action $(u, a) \to uau^*$. Clearly the center Z of A is the fixed point set of G.

A word about the relation of G to A is in order. The uniform (resp. weak) closure of the complex linear span of G is the C*-algebra (resp. von Neumann algebra) generated by A. In general, the unitary operators in A + iA need not include, nor be included in, the group G. In fact, it is easy to show that if either inclusion holds, then A + iA must be a von Neumann algebra.

By a *center-valued trace* we shall mean a map $\phi : A \to Z$ such that:

1) ϕ is linear.
2) $\phi(za) = z\phi(a)$ for $a \in A$, $z \in Z$.
3) $\phi(a) \geqq 0$ if $a \geqq 0$, $a \in A$.
4) $\phi(sas) = \phi(a)$ for $a \in A$, s a symmetry in A.
5) $\phi(1) = 1$.

From 2) and 5) we have $\phi(z) = z$, for all $z \in Z$. The notions of *normality* and *faithfulness* are defined as in [2] (p. 80). A familiar argument shows that a center-valued trace which is completely additive on projections has a largest "null projection" which, being central, must be zero. Such a trace, therefore, is automati-

cally faithful.

For $a \in A$, let K_a be the uniformly closed convex hull of the orbit of a (under the action of G).

LEMMA 31. Let A be a JW-algebra with a center-valued trace ϕ. Then for any $a \in A$, $K_a \cap Z$ contains at most one point.

PROOF. $K_a = K_{(a-\phi(a))+\phi(a)} = \phi(a) + K_{a-\phi(a)} \subset \phi(a) + N$, where $N =$ the null space of ϕ. Since $N \cap Z = 0$, $\phi(a)$ is the only point in $K_a \cap Z$, provided the latter is non-empty.

THEOREM 25 (THE APPROXIMATION THEOREM). If A is a JW-algebra, then $K_a \cap Z$ is non-empty, for each $a \in A$.

This is Dixmier's "Theoreme d'Approximation" [2] (Theoreme 1, p. 272). The facts needed here are contained in Lemmes 1, 2 and 3 [2] (pp. 269-271) and Theoreme 1 of [5] (p. 215). Generalized Comparability (for orthogonal projections) is needed for Lemme 1 [2] (p. 269).

COROLLARY 28. A JW-algebra has at most one center-valued trace.

This is immediate from Lemma 32 and Theorem 25.

At this point it is possible to exploit the excellent work of Dixmier on center-valued traces [2, 5] to achieve our goal, namely, existence of the trace for modular JW-algebras. This is hardly surprising, since Dixmier had already envisioned such applications of his work in [5].

THEOREM 26. A JW-algebra is modular if and only if it possesses a (unique) faithful normal center-valued trace.

The proof of the "if" part is an easy consequence of Corollary 9. A careful examination of [2] (Chapitre III, §8, sections 2 - 5) shows that Dixmier's proof of the deep "only if" part of this theorem for von Neumann algebras can be carried over to the present context with very minor modifications of phraseology.

19. A new kind of factor. In this section we shall construct a modular JW-factor of type I ("discrete, finite class") which is infinite dimensional as a real linear space.

By a spin system we shall mean a set P of symmetries $\neq \pm 1$ such that $st = -ts$ for $s, t \in P$ with $s \neq t$. We shall be concerned with infinite spin systems inside of a hyperfinite type II_I von Neumann factor. For simplicity, we suppose the

underlying Hilbert space to be separable. Von Neumann showed that the trace on a II_{I} factor is a finite sum of vector states and so is weakly and strongly continuous (see [6], Remark 2.1, for a simple proof).

Observe that any spin system must lie in the null space of the trace, since $tr(s) = tr(st^2) = -tr(tst) = -tr(s)$ if $s, t \in P$. Now let H be the weak closure of the real linear space spanned by P. Then H lies in the s.a. part of the null space of the trace. We now define a real inner product $[a, b]$ on the s.a. part of the II_{I} factor by setting $[a, b] = tr(a \circ b)$. Supplied with this structure, H is a real pre-Hilbert space with norm $\|a\|_2 = [a, a]^{1/2}$. Obviously the spin system P is an orthonormal set in H.

LEMMA 32. <u>If</u> $\{a_n\}$ <u>is a net of s.a. operators satisfying</u> $a_n^2 = \lambda_n$, λ_n <u>a real scalar</u> $\geqq 0$, <u>for each</u> n, <u>and if</u> $a_n \to a$ <u>strongly, then</u> $a_n^2 \to a^2$ <u>weakly and</u> $a^2 = \lambda$, <u>where</u> $\lambda = \lim \lambda_n$.

PROOF. If $a_n \to a$ strongly, then $a_n^2 \to a^2$ weakly by the proof of Lemma 1. Thus $(a_n^2 x | x) = \lambda_n (x | x) \to \lambda (x | x)$, for all vectors x, so $a^2 = \lambda$ because two bilinear forms agreeing on the diagonal agree.

COROLLARY 29. <u>Each non-zero operator in</u> H <u>is a positive multiple of a symmetry</u>.

PROOF. For $a \in H$, let $\{a_n\}$ be a net in $[P]$ (= the real linear span of P) converging strongly to a. To see that each a_n has the form $a_n^2 = \lambda_n$, consider a finite real linear combination $\sum_1^n \alpha_i s_i$ with α_i real and each $s_i \in P$. Then $(\sum_1^n \alpha_i s_i)^2 = \sum_1^n \alpha_i^2 + \sum_1^n \alpha_i \alpha_k (s_i \circ s_k) = \sum_1^n \alpha_i^2$. Hence $a^2 = \lambda$ by the Lemma, $\lambda \geqq 0$. If $\lambda = 0$, $a = 0$. For $\lambda > 0$, $s = \lambda^{-1/2} a$ is a symmetry.

COROLLARY 30. $A = R \oplus H$ (R = <u>all real multiples of</u> 1) <u>is a JW-algebra</u>.

PROOF. For $\alpha + \beta h \in R \oplus H$ we have $(\alpha + \beta h)^2 = (\alpha^2 + \beta^2 \gamma^2) + 2\alpha\beta h \in R \oplus H$, where $h = \gamma s$, s a symmetry, γ real and $\geqq 0$. Thus A is a Jordan algebra, weakly closed since H is.

LEMMA 33. <u>For</u> $a \in A = R \oplus H$ <u>with</u> $a = \alpha + \beta s$ <u>and</u> s <u>a symmetry in</u> H, <u>we have</u> $a \geqq 0$ <u>if and only if</u> $\|\beta s\|_2 = |\beta| \leqq \alpha$.

PROOF. First suppose $a \geqq 0$. We claim that $\alpha \geqq 0$. For if $\alpha < 0$, then $0 < -\alpha \leqq \beta s$, a contradiction, since a scalar multiple of a symmetry ($\neq \pm 1$) cannot be com-

parable with zero unless it is the zero multiple.

Now s is the difference of the two orthogonal complementary projections $e = (1+s)/2$ and $f = (1-s)/2$. Hence if $\alpha + \beta s \geqq 0$, then $(\alpha+\beta)e = e(\alpha+\beta s)e \geqq 0$ so that $\alpha + \beta \geqq 0$. Also $(\alpha - \beta) f = f(\alpha + \beta s) f \geqq 0$ so that $\alpha - \beta \geqq 0$. Thus $|\beta| \leqq \alpha$. Clearly $\|\beta s\|_2 = |\beta|$.

Conversely, if $\|\beta s\|_2$ $(= |\beta|) \leqq \alpha$ we claim that $\alpha + \beta s \geqq 0$. It is enough to show that $\alpha + \beta s$ has non-negative spectrum. Take any $\lambda < 0$ and consider $((\lambda - \alpha) - \beta s)^{-1}$. We shall show that this inverse exists and is of the form $\gamma + \partial s$ for suitable reals γ and ∂. In order to show that $1 = (\gamma + \partial s)((\lambda - \alpha) - \beta s) = (\gamma(\lambda - \alpha) - \partial\beta) + (\partial(\lambda - \alpha) - \gamma\beta)s$, we need only show that it is possible to solve the pair of linear equations:

$$\gamma(\lambda - \alpha) - \partial\beta = 1$$
$$\partial(\lambda - \alpha) = \gamma\beta$$

for γ and ∂ in terms of α, β and λ. But the determinant of this system is $-\beta^2 - (\lambda - \alpha)^2$ and this vanishes if and only if $\beta = 0$ and $\lambda = \alpha$. But by our assumption, $|\beta| \leqq \alpha$ so that $\lambda < 0 \leqq \alpha$, and these equations are always solvable.

We now digress momentarily to consider an example of a quantum mechanical system of bounded observables constructed by Lowdenslager ([16], Example 2, p. 90). Certain formal analogies with the JW-algebra $A = R \oplus H$ constructed above will be evident.

We reproduce Lowdenslager's example for the sake of completeness. Let H be a real pre-Hilbert space (Lowdenslager requires that H be complete, but this is inessential for our purposes) and S the space whose points are pairs (h, t) where $h \in H$, t is real, with the obvious linear operations. The positive cone S^+ is defined to be the set of (h, t) with $\|h\| \leqq t$, where $\|.\|$ is the norm of H. Lowdenslager proceeds to show that S ordered by S^+ is not a lattice. Quite the reverse is true, in fact, as we show in

PROPOSITION 21. Let H have dimension at least two. Then the partially ordered linear space S ordered by S^+ is an antilattice (i.e. two elements in S have a GLB there just in case they are comparable in the ordering).

REMARK. Geometrically, the antilattice condition means simply that two translates of the positive cone S^+ intersect in a translate of S^+ if and only if one of the translates contains the other.

PROOF. Given $a', b' \in S$ with GLB g'. Then $a = a' - g'$ and $b = b' - g'$ are $\geqq 0$ and have GLB zero. Hence we are reduced to showing that if $a, b \geqq 0$ with GLB $(a, b) = 0$, then $a = 0$ or $b = 0$.

Next note that for any $\alpha \geqq 1$, GLB$(\alpha a, b)$ exists and is zero. For if $x \leqq \alpha a$, b then $\alpha^{-1}x \leqq x$ so $\alpha^{-1}x \leqq a, b$. Thus $\alpha^{-1}x \leqq \mathrm{GLB}(a, b) = 0$ and $x \leqq 0$. Moreover, for any $\alpha > 0$, $\mathrm{GLB}(\alpha a, \alpha b) = \alpha\, \mathrm{GLB}(a, b)$. Thus $\mathrm{GLB}(\alpha a, \beta b) = 0$ if $\alpha, \beta > 0$ and $\mathrm{GLB}(a, b) = 0$.

Thus we are further reduced by normalization to the case where $a = (g, 1)$, $b = (h, 1)$ with $g, h \in H$ (both of norm $\leqq 1$). Observe that if either g or h is zero, say $g = 0$, then $(1 + \|h\|)a - b = (-h, \|h\|) \geqq 0$ and by the remarks above, $b = \mathrm{GLB}((1 + \|h\|)a, b) = 0$, proving the contention. We are assuming then that $g \neq 0 \neq h$ with GLB $(a, b) = 0$ and aiming for a contradiction.

We now assert that a and b can be assumed to lie on the "surface" of S^+ in the sense that $\|g\| = 1 = \|h\|$. For we may replace a by $a_1 = (g/\|g\|, 1)$ and b by $b_1 = (h/\|h\|, 1)$. Choose an integer $n > 1$ so that $n^{-1} \leqq \min(\|g\|, \|h\|)$. We then have $0 \leqq a_1 \leqq na$ and $0 \leqq b_1 \leqq nb$ with $\mathrm{GLB}(a_1, b_1) = 0$ by the remarks above. We have now reduced to the situation where $a = (g, 1)$, $b = (h, 1)$ with $\|g\| = 1 = \|h\|$ and $\mathrm{GLB}(a, b) = 0$.

The remainder of the argument falls into two parts. First we consider the case where g and h are antipodal ($g+h = 0$; this is essentially the situation treated by Lowdenslager). Choose a vector $p \in H$ with $p \perp g$, $\|p\| = 1$, using the assumption that H is at least two dimensional. Take $c = (p/2, 1-(3/2)^{1/2})$.

Then $a - c = (g - p/2, (3/2)^{1/2})$ and $\|g - p/2\|^2 = (g \mid g) - (g \mid p) + 1/4\,(p \mid p) = 1 - 0 + 1/4 = 5/4$, so $a - c \geqq 0$. Similarly, $b - c \geqq 0$. But $c \ngeqq 0$ since $\|p/2\| = 1/2 \nleqq 1-(3/2)^{1/2}$ and $c \nleqq 0$ because $1/2 \nleqq (3/2)^{1/2} - 1$.

Finally, if $g + h \neq 0$, set $p = (g+h)/\|g+h\|$ and let $c = (p/2, 1-(3/2)^{1/2})$ as before. Then $(p \mid h) = (p \mid h)$ and $\|p\| = 1$, so $(p \mid g) = \|g+h\|/2$. Since $\|g+h\| \leqq 2$, $0 \leqq 1 - \|g+h\|/2 \leqq 1$. Thus $\|g - p/2\|^2 = (1 - \|g+h\|/2) + 1/4 \leqq 1 + 1/4 = 5/4$ and again we conclude that $a - c \geqq 0$ and $b - c \geqq 0$. The inequalities above show that $c \ngeqq 0$ and $c \nleqq 0$. These conclusions however contradict our original assumption $0 = \mathrm{GLB}(a, b)$.

Kadison [9] has characterized von Neumann factors (within the class of von Neumann algebras) as the antilattices. With the Comparison Theorem (Theorem 10) at hand, we could reproduce the results of [9] for JW-algebras. Only the simpler

half of Kadison's argument, however, interests us here and we include the Jordan version for completeness.

PROPOSITION 22. If a JW-algebra is a antilattice, then it is a factor.

PROOF. Each pair of commuting s.a. operators a and b in a JW-algebra A have a GLB (namely $a \wedge b = (a+b-((a-b)^2)^{1/2})/2$) with respect to their commutant $\{a, b\}'$ relative to A. For if $c \in \{a, b\}'$ with $a, b \geq c$ we have $c \in M$, where M is some maximal abelian subalgebra of A containing a and b. Hence $c \leq a \wedge b$, since $a \wedge b \in M$. Now the center Z of A has a functional representation as a real $C(X)$ for X compact (X = the pure state space of Z). But the commutant in A, $\{a, b\}'$ of $a, b \in Z$ is A, since $a, b \in M$ for all maximal abelian subalgebras M of A. Because A is an antilattice, Z is totally ordered, so X has one point and Z is the real field.

Getting back now to our concrete JW-algebra $A = R \oplus H$ of Corollary 30 we see from Proposition 21 that this algebra is an antilattice so that A is a factor by Proposition 22. We shall refer to such algebras as spin factors. It is not hard to describe the maximal abelian subalgebras of A. They are just the planes passing through the scalar axis of the positive cone.

We have observed that H lies entirely in the (weakly closed) null space of the trace, so that every symmetry in H has trace zero. From the 1-1 correspondence between symmetries and projections (given by $s = 2e - 1$ and $e = (1 + s)/2$), it follows at once that all projections in A different from 0 and 1 lie in the hyperplane $\{a \in A: \operatorname{tr}(a) = 1/2\}$ and that the dimension function gotten by restricting the trace to the projection lattice of A takes three values: 0, $1/2$ and 1. In particular, each projection $(\neq 0, 1)$ is both maximal and minimal, therefore abelian and faithful so that A is type I. Any two such projections are exchanged by a symmetry in A (Corollary 26).

If the spin system P is infinite (there are "infinitely many degrees of freedom"), then the spin factor constructed from it is necessarily infinite dimensional. Now a hyperfinite II_{I} von Neumann factor always contains a countably infinite spin system (which generates it as a von Neumann algebra, in fact). We summarize the results of this section in

THEOREM 27. Infinite dimensional type I modular JW-factors exist.

Infinite spin systems are intimately related to skew distributions and Clifford algebras over a Hilbert space. For a thorough discussion of these matters we refer the reader to [19]. Jordan, von Neumann and Wigner determined all finite dimensional type I (modular) JW-factors in [8]. These are i) all n x n complex s.a. matrices; ii) all n x n real symmetric matrices; iii) finite dimensional spin factors; and iv) all n x n ($n \geq 3$) s.a. matrices with quaternion entries.

20. Concluding remarks. To the reader familiar with Kaplansky's theory of AW*-algebras [10, 11], it will be clear that much of the theory we have developed goes through under an assumption short of weak closure. Very little modification of the proofs given in this paper is needed to parallel the AW*-theory. One need only define an AJW-algebra to be a uniformly closed Jordan algebra of s.a. operators satisfying (A) Any set of orthogonal projections has a LUB (this replaces Theorem 1), and (B) Each maximal abelian subalgebra is uniformly generated by its projections (this replaces Lemma 2).

Theorem 26 must be scrapped (this is still open for AW*-algebras).

Theorem 15 needs to be modified as in [11] (Theorem 4, p. 471).

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